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GEOMETRICAL AND DIFFUSIVE EFFECTS IN NONLINEAR ACOUSTIC PROPAGATION OVER LONG RANGES

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CONTENTS

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This paper gives asymptotic solutions to generalized Burgers' equations governing the propagation of weakly nonlinear acoustic waves under the influence of geometrical spreading and thermoviscous diffusion. Geometrical effects are included through a general ray-tube area function, $\mathscr{A}(r)$, and solutions calculated to arbitrarily large ranges, for both N-wave and sinusoidal initial signals, in two limits, one in which a dimensionless diffusivity is allowed to vanish for fixed values of source Mach number and spreading parameter $\omega r_0/a_0$, the other in which the source Mach number is allowed to increase indefinitely for fixed values of the other parameters, the latter simulating experiments in which the phenomenon of amplitude saturation was investigated. At moderate ranges the wave pattern comprises the lossless portions, separated by shocks of Taylor structure, of weak-shock theory, and we pinpoint a number of non-uniformities in this description at large ranges, rectifying the non-uniformities with appropriate new asymptotic descriptions of the wave and classifying the area functions, $\mathscr{A}(r)$, accordingly. The validity of weak-shock theory at large ranges is delineated, and, where it fails, different routes to the final old-age linear decay are identified. Complete solutions in old age are obtained for a broad class of area functions including that for spherical waves. Amplitude saturation in the appropriate limit is shown to be a general phenomenon for these model equations. The paper ends with a discussion of various *ad hoc* approaches to approximate solution of generalized Burgers' equations and with some comparison with experiment.

1. INTRODUCTION

This paper is concerned with asymptotic solutions to generalized Burgers' equations (GBE) of the form

$$\frac{\partial u}{\partial r}-\frac{\gamma+1}{2a_0^2}u\frac{\partial u}{\partial \tau}+\tfrac{1}{2}u\frac{\mathrm{d}}{\mathrm{d}r}\ln\mathscr{A}(r)=\frac{\delta}{2a_0^3}\frac{\partial^2 u}{\partial \tau^2} \tag{1.1}$$

$$u(r_0,\tau)=u_0 f(\omega\tau). \tag{1.2}$$

Equations of this form describe the propagation of weakly nonlinear longitudinal waves in a gas or liquid, subject not only to the diffusive effects associated with viscosity and thermal conductivity represented by the term on the right side, but subject to the geometrical effects of change of the 'ray tube area' $\mathscr{A}(r)$ represented by the last term on the left. In (1.1) $u(r,\tau)$ is the particle velocity along a ray tube, a_0 is the small signal sound speed, r is range, $\tau = t-(r-r_0)/a_0$ is retarded time, δ Lighthill's (1956) diffusivity of sound and γ the specific heat ratio for a gas or an empirically determined constant for a condensed phase ($\gamma \approx 7$ for water). Equation (1.1) is written in a form convenient for signalling problems, in which u is prescribed as a function of t at $r = r_0$, and therefore r must be regarded as a time-like variable, τ as a (space-like) phase variable. Important examples of (1.1) cover freely diverging spherical or cylindrical waves, which have as the last term on the left, u/r and $u/2r$, respectively, whereas $\tfrac{1}{2}ku$ corresponds to the case of propagation in an exponential horn of flare constant k, $\mathscr{A} = \mathscr{A}_0 \exp(kr)$. The GBE for spherical and cylindrical waves was derived by Leibovich & Seebass (1974, p. 117) by using a formal multiple-scales method which can readily be extended to general $\mathscr{A}(r)$, and in fact the derivation of (1.1) is sketched by Lighthill (1956, section 9.1); the reader is referred to these articles for details.

For plane flow, $\mathscr{A}$ = const., (1.1) is the ordinary Burgers' equation, whose exact solution via the Cole–Hopf transformation has been much studied (see, for example, Lighthill 1956; Benton & Platzman 1972; Whitham 1974; Rudenko & Soluyan 1977). From these studies it is well known that shocks form in any wave that has a compressive phase, and that for moderate ranges beyond the shock-formation range the shocks have an interior structure, which is resolved by thermoviscous diffusion and described by the hyperbolic tangent solution of Taylor (1910); outside the shocks, whose location can, at these ranges, be predicted by 'weak-shock theory' (Whitham 1974), the flow is lossless. From the Taylor solution it can be seen that the shock thickness increases with range, ultimately becoming of the same scale as the whole wave, at which stage the shock amplitudes have been so much reduced by dissipation within the shocks that linear dynamics prevails over the whole wave which then subsides in 'old age'. Although it is not obvious from the exact solution, the shock thickening simultaneously makes the Taylor solution invalid as a leading order description of the shock structure, and in some problems there may also be a simultaneous non-uniformity in the outer lossless solution. A further possible non-uniformity may also arise at the same typical range. In this the shock centre begins to drift, under thermoviscous forces, far from its weak-shock theory location (Lighthill 1956, p. 337), leading to a distortion of the shock profile which cannot be described by perturbations away from the Taylor solution. As, in the plane-wave case, some of these non-uniformities arise at a larger range than that associated with shock thickening, a failure to recognize them is not crucial and it is usual to regard decay into old age as associated primarily with the shock-thickening mechanism.

When $\mathscr{A}(r)$ is not constant, however, these non-uniformities must be taken much more seriously. The effect of diverging or converging geometry is often to separate out the ranges at which various non-uniformities occur, and the new possibilities may arise before the final shock thickening, if indeed that ever occurs. Some of these sequences of non-uniformity were studied by Crighton & Scott (1979) for the particular cases of cylindrical and spherical geometry, where matched asymptotic expansion methods (based on the smallness of an appropriate dimensionless δ) were used to describe the entire evolution and decay of N-waves and sinusoidal waves. Such asymptotic methods, or numerical methods, are needed because of the *non-existence* of Bäcklund transformations (and, therefore, of generalizations of the Cole–Hopf linearization) for any equation of the class (1.1) except that for plane flow ($\mathscr{A}$ = const.) (Nimmo & Crighton 1982). Indeed, the only exact solution known for non-planar flow is a similarity solution for cylindrical waves (Rudenko & Soluyan 1977, p. 70). The matched expansion approach runs up against severe difficulties in that at some stage the full GBE (1.1), with all terms comparable, has to be solved. In some cases, when the region involved does not extend across the whole wave, it is possible to side-step the difficulty of finding an exact solution to (1.1), and then it is possible to follow the wave development to arbitrarily large ranges and to get not only the functional form of the old-age solution but also precise details of numerical amplitude factors. This was found to be the case for spherical waves. For cylindrical waves, on the other hand, the GBE holds across the whole wave and although it is still possible to find the functional form of the subsequent decay it is not possible to achieve the same numerical detail of the amplitude. Even then, however, the matched expansion approach still offers an advantage in that it reduces the number of parameters in the full GBE problem to one.

The aim of this paper is to extend the work of Crighton & Scott (1979) to cover all geometric effects described by the class (1.1). We study, again by matched expansion methods, a dimensionless version of (1.1) in which two parameters, ϵ, z_0 occur, and in the first place we take the limit $\epsilon \rightarrow 0, z_0 = O(1)$, corresponding to a fixed balance between nonlinear and geometric effects with one infinitesimal nominal ratio of diffusive to nonlinear effects. As before, two particular signalling conditions will be adopted, corresponding to N-waves and to sinusoidal waves at the 'source' $r = r_0$. The results for these are thought to be representative of a much wider class of conditions, and although special initial conditions could be chosen so as to produce qualitatively different results from those below, these would not be thought to be generic. We intend to identify all types of leading-order non-uniformity for (1.1) with these signalling conditions and to categorize accordingly the area functions, $\mathscr{A}(r)$. It will be found that the area functions may be divided into a relatively small number of subclasses according to the nature of the non-uniformity (or *sequence* of non-uniformities) to which they give rise and which then determine the mode of transition to 'old age' (if it exists). These results will, naturally, determine the spatial extent and parameter range for which weak shock theory remains valid in a horn of given cross section $\mathscr{A}(r)$.

In a second limit, we take $\epsilon \rightarrow 0$ with $\epsilon z_0 = O(1)$. This limit is intended to model an experimental situation in which the diffusive and geometrical parameters are kept fixed and the 'source amplitude' steadily increased. In the plane-wave case this is known (see, for example, Rudenko & Soluyan 1977, p. 49) to lead to the phenomenon of 'amplitude saturation', the non-dependence of old-age amplitude (and indeed the amplitude of the sawtooth signal at smaller ranges) on the source amplitude. This is an important practical phenomenon and it is of interest to ascertain the extent to which it emerges in the presence of area variations. We prove here that, provided that $\mathscr{A}(r)$ is such that shocks can form, saturation is a general phenomenon.

There have been many previous attempts to find approximate solutions to generalized Burgers' equations, for spherical waves in particular, by numerical methods and by methods based on various *ad hoc* physical arguments. On the numerical side we should mention in particular the paper by Sachdev & Seebass (1973) in which the evolution of cylindrical and spherical N-waves is calculated; here, however, the evolution was not followed to sufficiently large ranges for the breakdown of the Taylor-shock structure to be seen, nor of course the ultimate linear decay. In a very recent extension of this work, Sachdev *et al.* (1986) use a mixture of spectral and finite difference methods to deal with such large ranges, and their work appears generally to agree with predictions for spherical N-waves given in Crighton & Scott (1979) and to supplement those predictions for cylindrical N-waves where the matched asymptotic solution is less complete. Reference must also be made to two papers by Enflo (1981, 1985). These papers deal, respectively, with cylindrical N-waves and cylindrical sinusoidal waves, problems in which the matched expansion approach of Crighton & Scott (1979) runs into difficulties when the full GBE (1.1) (with all coefficients scaled to unity) is found to hold over the whole wave when the long-range non-uniformity sets in. Enflo attempts to solve this by perturbation away from the linear old-age solution with an undetermined amplitude coefficient C. This is precisely the technique which was used successfully by Fay (1931), who was able, in the plane-wave case, to sum the perturbation series and obtain a uniformly valid approximation (which indeed turned out to be an exact and very important solution of the ordinary Burgers' equation). In the plane-wave case one can thus determine C by matching to either the lossless outer solution or to the inner Taylor shock. For cylindrical N-waves this approach fails, because the matching to the lossless solution, while possible, is trivial and does not determine C, while matching to the shock cannot be accomplished as the perturbation series cannot be summed in the appropriate space–time region ($\tau^2/r = O(1)$ is needed, in dimensionless form, whereas the series appears to be good only for τ^2/r large). A comparison between Enflo's prediction of C and the computations of Sachdev *et al.* would be interesting none the less. For sinusoidal cylindrical waves, Enflo (1985) finds certain asymptotics of each of the terms in the perturbation series and obtains a partial summation of a large finite number of these asymptotic expressions, claiming that this sum is valid in a narrow region which overlaps with part of the Taylor shock. Matching then in principle determines C, but actually yields only an *estimate*, because, in contrast to Fay's plane-wave solution, only a partial resummation of the perturbation series was carried out.

Other approximate approaches to the GBE are discussed in §10. In general, these do not refer specifically to the form of dissipation present in (1.1) and, for sinusoidal waves, can also be applied to systems in which the dissipation is replaced by a term simply proportional to u. For such systems an exact solution of the full equation is possible, and these exact solutions are given in §10 and compared with the predictions of approximate methods due to Rudnick (1953) and Shooter *et al.* (1974). These approximate predictions are also compared with our own prediction wherever possible. In some cases remarkable agreement in functional form is found and, in one case, also even precise agreement of coefficients, so that these approximate methods may serve as rough guides to the physical balances involved. Some comparisons with experimental data on amplitude saturation for plane waves in tubes and for spherically spreading waves are also included in §10.

Appendixes contain some mathematical details whose inclusion in the main text would unduly interrupt the main line of argument.

2. The generalized Burgers' equation: equivalent non-dimensional forms

From the various dimensional quantities appearing in the boundary value problem defined by (1.1) and (1.2), we define the two dimensionless parameters

$$\epsilon = \delta\omega/a_0 u_0(\gamma+1) \quad \text{and} \quad z_0 = \tfrac{1}{2}(\gamma+1)\, u_0 r_0 \omega/a_0^2. \tag{2.1}$$

We shall first of all consider the problem (1.1 and 1.2) in the small diffusivity limit in which $\epsilon \to 0+$ and z_0 has a fixed $O(1)$ value. In this case we define the non-dimensional variables

$$U = u/u_0, \quad R = r/r_0,$$
$$\theta = \omega\tau, \quad A(R) = \mathscr{A}(r)/\mathscr{A}(r_0), \tag{2.2}$$

so that (1.1) and (1.2) become

$$U_R - z_0\, UU_\theta + \tfrac{1}{2}(A'/A)\,(R)\,U = \epsilon z_0\, U_{\theta\theta} \tag{2.3}$$

and

$$U(1,\theta) = f(\theta). \tag{2.4}$$

For $R = O(1)$, however, this form of the generalized Burgers' equation is not very convenient, and instead we shall study the equivalent problem

$$W_z - WW_\theta = \epsilon a(z)\, W_{\theta\theta}, \tag{2.5}$$

with
$$W(0,\theta) = f(\theta) \tag{2.6}$$
and
$$a(z) = A^{\frac{1}{2}}(R).$$
Also, for larger R, it will be most convenient to consider the problem in the second alternative form
$$W_R - A^{-\frac{1}{2}}(R)\, W W_\theta = \epsilon W_{\theta\theta}, \tag{2.7}$$
where in (2.5)–(2.7) $W = A^{\frac{1}{2}}(R)\, U$ is a non-dimensional velocity from which linear effects due to area changes have been removed, and
$$z = z_0 \int_1^R A^{-\frac{1}{2}}(R')\, \mathrm{d}R' \equiv z_\infty I_A(R), \tag{2.8}$$
where $z_\infty = z_0$ if $\lim_{R\to\infty} I_A(R) = \infty$, and is such that $\lim_{R\to\infty} I_A(R) = 1$ otherwise.

Later, in order to investigate the possibility of amplitude saturation for different area functions, we shall consider the problem in which the source amplitude u_0 is indefinitely increased at constant values of all the other quantities; to achieve this we study (1.1, 1.2) in the limit as ϵ tends to zero from above and with ϵz_0 given a fixed $O(1)$ value.

Thus for these purposes we take
$$U = u/u_0, \quad R = r/r_0, \quad \theta = \omega\tau,$$
$$A(R) = \mathscr{A}(r)/\mathscr{A}(r_0), \quad z_0 = (\delta\omega^2 r_0/2a_0^2)\,\epsilon^{-1} = c\epsilon^{-1},$$
and
$$W = A^{\frac{1}{2}}(R)\, U, \tag{2.9}$$
so that
$$W_z - W W_\theta = a(\epsilon, z)\, W_{\theta\theta}, \tag{2.10}$$
where
$$a(\epsilon, z) = \epsilon A^{\frac{1}{2}}(R) \quad \text{and} \quad z = c\epsilon^{-1} \int_1^R A^{-\frac{1}{2}}(R')\, \mathrm{d}R'. \tag{2.11}$$

These problems have previously been studied by Crighton & Scott (1979) (hereafter referred to as C.S.) for the special cases of plane, cylindrical and spherical Burgers' equations, corresponding to $A(R) = 1, R, R^2$, respectively. Also, Scott (1981 *a*) has considered the problem defined by (1.1) with sinusoidal boundary condition, in the whole of ϵ, z_0-parameter space, for cylindrical and spherical Burgers' equations.

In the following sections we shall consider (2.1 and 2.2) in the limit $\epsilon \to 0+$ and $z_0 = O(1)$.

3. The solution for $R = O(1)$

In Appendix I we shall consider the boundary value problem defined by (2.5 and 2.6), with $f(\theta)$ a sufficiently smooth but otherwise arbitrary function, and obtain matched two-term inner and outer solutions, the 'lossless' solution and its $O(\epsilon)$ correction, together with the Taylor-shock solution (Taylor 1910) and its $O(\epsilon)$ correction. For the time being, however, we shall simply quote these expressions for the two functions $f(\theta)$ that we will consider in this paper; (*a*) harmonic $f(\theta)$ and (*b*) N-wave $f(\theta)$. For brevity we shall refer to the description of the solution in terms of these matched inner and outer expansions as the 'order one composite'.

(*a*) *Harmonic* $f(\theta)$

We have $f(\theta) = \sin\theta$ and, by Appendix I, we find the outer solution, $W = W_0(z,\theta) + \epsilon W_1(z,\theta) + o(\epsilon)$, where
$$W_0(z,\theta) = \sin p, \quad \theta = p - z\sin p \tag{3.1}$$
and
$$W_1(z,\theta) = -\frac{\sin p}{1 - z\cos p} \int_0^z \frac{a(z')\,\mathrm{d}z'}{(1 - z'\cos p)^2}. \tag{3.2}$$

Shocks are inserted for $z > 1$ at $\theta = \pm 2n\pi, n = 0, 1, \ldots$, and without loss of generality we consider only the shock at $\theta = 0$. For this shock we find that the inner solution is given by (AI 3, AI 4) with

$$h^{\pm}(z) = \pm \sin p_0, \quad \text{where} \quad p_0 = z \sin p_0,$$

$$c(z) = 0 \quad \text{and} \quad K(z) = \frac{a'}{h} \ln 2 - \frac{1}{2} \frac{h}{1 - z \cos p_0} \int_0^z \frac{a(z')\,dz'}{(1 - z' \cos p_0)^2}.$$

(b) N-wave $f(\theta)$

In this case $f(\theta) = \{^{-\theta \;\; |\theta| < 1}_{0 \;\;\;\; |\theta| > 1}$ and from C.S. we have $W = W_0(z, \theta) + o(\epsilon^n)$ for all n, where

$$W_0(z, \theta) = \begin{cases} -\theta/(1+z), & |\theta| < (1+z)^{\frac{1}{2}}, \\ 0, & |\theta| > (1+z)^{\frac{1}{2}}, \end{cases} \tag{3.3}$$

with shocks at $\theta = \pm (1+z)^{\frac{1}{2}}$. We concentrate our attention on the shock at $\theta = +(1+z)^{\frac{1}{2}}$, where we have

$$h^{+}(z) = 0, \quad h^{-}(z) = -(1+z)^{-\frac{1}{2}}, \quad c(z) = -(1+z)^{\frac{1}{2}} \int_0^z \frac{a(z')\,dz'}{(1+z')}$$

and

$$K(z) = 4(1+z)^{-\frac{1}{2}} a' - \tfrac{1}{2}(1+z)^{-\frac{1}{2}} \int_0^z \frac{a(z')\,dz'}{(1+z')},$$

and consequently the inner expansion from (AI 3, AI 4).

This form of the solution, the set of matched asymptotic expansions hereafter referred to as the order one composite, is valid for $R = O(1)$ and for some larger ranges. However, for most area functions $A(R)$, there comes a range where one or more of the assumptions of weak-shock theory, upon which this solution is based, becomes no longer valid and we must find another form for the solution. In the following section we shall see how the nature of this breakdown of weak-shock theory depends on the particular area function appearing in the generalized Burgers' equation (2.1).

4. Breakdown of the order-one composite

A breakdown of the kind referred to above and in §1 will occur for one or more of the following three reasons.

(A) The predicted thickness of the Taylor shock becomes of the same order as the scale, $m(z)$, of the whole wave; this comes about when z is such that

$$T \equiv O[a(z)/(h(z)\,m(z))] = O(1). \tag{4.1}$$

(B) The 'correction due to diffusivity' (Lighthill 1956 and C.S.) displaces the shock centre a 'relatively large' distance from the location $\theta_s(z)$ predicted by the equal areas rule of weak shock theory; this occurs if

$$S \equiv O[\epsilon c(z)/\theta_s(z)] = O(1). \tag{4.2}$$

(C) The Taylor-shock solution $W_0^*(z, \theta^*)$ fails to be a uniformly valid $O(1)$ representation of the shock, and this occurs when

$$N \equiv O\left(\epsilon \frac{W_1^*}{W_0^*}\right) = O\left(\epsilon \frac{h'(z)\,a(z)}{h^3(z)}, \epsilon \frac{a'(z)}{h^2(z)}, \epsilon \frac{k(z)}{h(z)}\right) = O(1). \tag{4.3}$$

A fourth possibility is that the *lossless* solution becomes non-uniform. For harmonic waves we find that this comes about at range $R = O(\epsilon^{-1})$, irrespective of area function, and for N-waves that such a non-uniformity never takes place. In all the cases we shall discuss below, the lossless solution non-uniformity will never arise before any of the other kinds of non-uniformity has occurred, and therefore, for simplicity, we shall, in this section, suppress further reference to it. Later, however, it will be seen that, in a particular case, this factor does indeed determine a breakdown of the form of the solution *subsequent* to the one currently under consideration.

It is now most convenient to express the three measures of non-uniformity in terms of the *original* dimensionless range variable, R, instead of the transformed variable z; in terms of R we have

$$T = O\left(\epsilon \frac{A^{\frac{1}{2}}(R)}{H(R)\, M(R)}\right), \tag{4.4}$$

$$S = O\left(\epsilon \frac{C(R)}{\theta_s(R)}\right), \tag{4.5}$$

$$N = O\left(\epsilon \frac{H'(R)\, A(R)}{H^3(R)}, \epsilon \frac{A'(R)}{H^2(R)}, \epsilon \frac{K(R)}{H(R)}\right), \tag{4.6}$$

where $H(R) = h(z)$, $M(R) = m(z)$, $\theta_s(R) = \theta_s(z)$, $C(R) = c(z)$ and $K(R) = k(z)$. Before proceeding further we shall now give a more precise definition of what we mean by an 'area function' $A(R)$.

We say that $A(R)$ is an area function if it is a strictly positive differentiable function which behaves in a monotonic fashion (increasing or decreasing) for large R. This definition is useful in the subsequent analysis, but is also descriptive of the sort of area changes one might expect in a physical horn. We define the set of all such area functions to be A, and we also define the following subsets of A which will be used in the classification of area functions which follows.

We define the following sets:

$$\mathrm{A}^C = \{A \in \mathrm{A} \mid I_A(R) \to 1\}, \tag{4.7}$$

the set of area functions, A, for which the integral $I_A(R)$ converges as $R \to \infty$;

$$\mathrm{A}^D = \mathrm{A} \backslash \mathrm{A}^C, \tag{4.8}$$

the complement of A^C in A, the set of area functions, A, for which the integral $I_A(R)$ diverges as $R \to \infty$;

$$\mathrm{A}^- = \{A \in \mathrm{A} \mid A(R) \to 0\}, \tag{4.9}$$

the set of 'converging' area functions and

$$\mathrm{A}^+ = \mathrm{A} \backslash \mathrm{A}^-, \tag{4.10}$$

its complement in A;

$$\mathrm{A}_{\mathrm{e}^-} = \{A \in \mathrm{A} \mid (A'/A)\,(R) \to \beta \in [-1, 0)\}, \tag{4.11}$$

the set of area functions which decrease exponentially quickly, or faster, as $R \to \infty$, for example $A \sim \mathrm{e}^{-\alpha R}$ or $A \sim \mathrm{e}^{-R^2}$;

$$\mathrm{A}_{\mathrm{pe}^-} = \{A \in \mathrm{A} \mid (A'/A)\,(R) \to 0, RA'/A \to -\infty\}, \tag{4.12}$$

the set of ('pseudo-exponential') area functions which decrease more slowly than the function

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$e^{-\alpha R}$, for any $\alpha > 0$, but more quickly than $R^{-\beta}$, for any $\beta > 0$; e.g. $A \sim e^{-R^{\frac{1}{2}}}$; for each real number λ we define

$$\mathrm{A}_\lambda = \{A \in \mathrm{A} \,|\, (RA'/A)\,(R) \to \lambda \in \mathbb{R}\}, \tag{4.13}$$

the set of area functions asymptotically different from the function R^λ by less than algebraic amounts (e.g. $A \sim R^\lambda, A \sim R^\lambda \ln R$ or $A \sim R^\lambda B(R)$, where $(B'/B)\,(R) \to 0$); related to A_λ we define

$$\mathrm{A}_\lambda^- = \{A \in \mathrm{A}_\lambda \,|\, A(R)/R^\lambda \to 0\}, \tag{4.14}$$

the set of area functions in A_λ which increase less rapidly than R^λ (e.g. $A \sim R^\lambda (\ln R)^{-1}$) and

$$\mathrm{A}_\lambda^+ = \mathrm{A}_\lambda \backslash \mathrm{A}_\lambda^-, \tag{4.15}$$

its complement in A_λ; for $\lambda = 2$ we define

$$\mathrm{A}_2^C = \mathrm{A}_2 \cap \mathrm{A}^C \quad \text{and} \quad \mathrm{A}_2^D = \mathrm{A}_2 \cap \mathrm{A}^D, \tag{4.16}$$

the sets of area functions in A_2 for which the integrals $I_A(R)$ converge and diverge, respectively, as $R \to \infty$ (e.g. $A \sim R^2(\ln R)^4 \in \mathrm{A}_2^C$, and $A \sim R^2(\ln R)^2 \in \mathrm{A}_2^D$); finally we define

$$\mathrm{A}_{\mathrm{e}^+} = \{A \in \mathrm{A} \,|\, (RA'/A)\,(R) \to \infty\}, \tag{4.17}$$

the set of area functions which increase more quickly than the function R^β for any $\beta > 0$, e.g. $A \sim e^{R^{\frac{1}{2}}}$ or $A \sim e^R$. Furthermore, because by the definition of area function we exclude functions which oscillate as $R \to \infty$, the above subsets must exhaust A; in fact we have

$$\mathrm{A} = \mathrm{A}_{\mathrm{e}^-} \cup \mathrm{A}_{\mathrm{pe}^-} \bigcup_{\lambda \in \mathbb{R}} \mathrm{A}_\lambda \cup \mathrm{A}_{\mathrm{e}^+}. \tag{4.18}$$

For the rest of this section we shall consider the reasons for the breakdown of the order one composite for harmonic and N-waves separately.

(a) *Harmonic waves*

In this subsection it will be shown that, for harmonic waves, the nature of the breakdown of the (Taylor shocks plus lossless arcs) form of the solution, depends on which of the five sets $(\mathrm{A}_{\mathrm{e}^-}; \mathrm{A}_{\mathrm{pe}^-}; \cup_{\lambda<2} \mathrm{A}_\lambda; \mathrm{A}_2^D; \mathrm{A}^C)$ the area function belongs to.

For harmonic waves, we have $h(z) \sim \frac{1}{2}\pi$ and $m(z) = 1$, and hence

$$\begin{gathered} T = O(\epsilon A^{\frac{1}{2}}(R)\, I_A(R)), \\ N = O[\epsilon A^{\frac{1}{2}}(R)\, I_A(R),\, \epsilon A'(R)\, I_A^2(R),\, \epsilon I_A^{-1}(R)\, I_{I_A^4}(R)]. \end{gathered} \tag{4.19}$$

Because $c(z) = 0$, by symmetry, there is no possibility of a non-uniformity of kind (B) defined above. Now we consider the five sets mentioned above individually.

(i) $A \in \mathrm{A}_{\mathrm{e}^-}$

Here we have $T = N = O(\epsilon A^{\frac{1}{2}} I_A)$, and because $\lim_{R\to\infty} (I_A/A^{-\frac{1}{2}}) = -2 \lim_{R\to\infty} A(R)/A'(R) \neq \infty$, by the defining property of $\mathrm{A}_{\mathrm{e}^-}$, we see that none of the measures of non-uniformity becomes $O(1)$. Therefore there is no breakdown in the $R = O(1)$ structure at larger ranges and hence *the order one composite remains uniformly valid* for all R.

(ii) $A \in \mathrm{A}_{\mathrm{pe}}$-

Once again $T = N = O(\epsilon A^{\frac{1}{2}} I_A)$, but this time $A'/A \to 0$ and so a non-uniformity must occur. This non-uniformity involves the whole of the wave, since T and N both become $O(1)$ at the same range, $R = O(R_1(\epsilon))$, where R_1 is given by

$$\epsilon A^{\frac{1}{2}}(R_1)\, I_A(R_1) = 1.$$

A non-uniformity of this kind is referred to as a 'gross' non-uniformity.

(iii) $A \in \cup_{\lambda<2} \mathrm{A}_\lambda$

For such area functions it follows, from the asymptotic forms of A', I_A and $I_{I_A^4}$ found in Appendix II, that we have from (4.19),

$$T = O(\epsilon R); \quad N = O\left(\epsilon R, \epsilon \begin{Bmatrix} R^{-2} A^{\frac{3}{2}}(R) & \lambda \neq 1 \\ R^{-2} \ln R A^{\frac{3}{2}}(R) & \lambda = 1 \end{Bmatrix}\right).$$

It is easy to see that because $A/R^2 \to 0$, T and N each become $O(1)$ at a typical range $R = O(R_1(\epsilon))$, where $\epsilon R_1 = 1$, and once more we have a gross non-uniformity.

(iv) $A \in \mathrm{A}_2^D$

Here, we have

$$T = O[\epsilon R \ln R], \quad N = O[\epsilon R \ln R, \epsilon R(\ln R)^2, \epsilon R^{-2}(\ln R)^{-2} A^{\frac{3}{2}}(R)]$$

and so we see that N becomes $O(1)$, while T is still $o(1)$, at range $R = O(R_1(\epsilon))$, where now

$$\epsilon R_1 (\ln R_1)^2 = 1.$$

This kind of breakdown, in which the non-uniformity is restricted to a thin shock region, is referred to as a 'localized' non-uniformity.

(v) $A \in \mathrm{A}^C$

In this case we find that

$$T = O[\epsilon A^{\frac{1}{2}}(R)], \quad N = O[\epsilon A^{\frac{1}{2}}(R), \epsilon A'(R), \epsilon R]$$

and so, because $A^{\frac{1}{2}}/R, A'/A^{\frac{1}{2}} \to \infty$, we see that $A'(R)$ becomes $O(\epsilon^{-1})$ before both $A(R)$ and R itself, so that the non-uniformity is one in which N becomes $O(1)$ at range $R = O(R_1(\epsilon))$, where

$$\epsilon A'(R_1) = 1,$$

and hence we have a localized non-uniformity.

This completes the study of the breakdown of the order one composite for harmonic waves. Next we shall consider the same issues for N-waves.

(b) N-waves

For N-waves the nature of the breakdown again divides A into five sets: $\mathrm{A}^-, \mathrm{A}_0^+, \cup_{0<\lambda<2} \mathrm{A}_\lambda, \mathrm{A}_2^D$ and A^C. In this case $h(z) \sim z^{-\frac{1}{2}}$ and $m(z) \sim z^{\frac{1}{2}}$, so that

$$T = O[\epsilon A^{\frac{1}{2}}(R)], \quad S = O[\epsilon I_{I_A^2}(R)], \quad N = O[\epsilon A^{\frac{1}{2}}(R), \epsilon A'(R)\, I_A(R), \epsilon I_{I_A^2}(R)]. \qquad (4.20)$$

(i) $A \in \mathrm{A}^-$

In this case T, S and N all remain $o(1)$ for all R and so the order one composite remains uniformly valid, even to arbitrarily large ranges.

(ii) $A \in \mathrm{A}_0^+$

Here, by (4.20) and Appendix II, we have

$$T = O[\epsilon A^{\frac{1}{2}}(R)], \quad S = O[\epsilon(\ln R)\, A^{\frac{1}{2}}(R)], \quad N = O[\epsilon A^{\frac{1}{2}}(R), \epsilon(\ln R)\, A^{\frac{1}{2}}(R)],$$

and hence S and N both become $O(1)$ at range $R = O[R_1(\epsilon)]$, where

$$\epsilon(\ln R_1)\, A^{\frac{1}{2}}(R_1) = 1.$$

This type of non-uniformity, in which the shock remains thin but is displaced to an unknown location, is called a 'translational' non-uniformity.

(iii) $A \in \bigcup_{0<\lambda<2} \mathrm{A}_\lambda$

In this case T, S and N are all $O(\epsilon A^{\frac{1}{2}}(R))$, and all consequently $O(1)$ at range $R = O(R_1(\epsilon))$, where here

$$\epsilon A^{\frac{1}{2}}(R_1) = 1,$$

provoking a gross non-uniformity.

(iv) $A \in \mathrm{A}_2^D$

Now we have $T = O(\epsilon A^{\frac{1}{2}}(R))$, $S = O[\epsilon(\ln R)^{-1} A^{\frac{1}{2}}(R)]$, $N = O[\epsilon A^{\frac{1}{2}}(R),\ \epsilon(\ln R)\, A^{\frac{1}{2}}(R),\ \epsilon(\ln R)^{-1} A^{\frac{1}{2}}(R)]$, and so a localized non-uniformity arises at range $R = O(R_1(\epsilon))$, where

$$\epsilon(\ln R_1)\, A^{\frac{1}{2}}(R_1) = 1.$$

(v) $A \in \mathrm{A}^C$

This is the same as case (v) for harmonic waves and once more we have a localized non-uniformity at range $R = O[R_1(\epsilon)]$, where $\epsilon A'(R_1) = 1$.

Thus we have categorized the area functions according to how, and at what range, non-uniformities arise in the order one composite. In the following three sections we shall find the forms of solution which follow the onset of a non-uniformity of localized, gross or translational type.

5. Scalings for the nonlinear region

For all area functions we will show that the motion in the region following the non-uniformity discussed above will be governed by generalized Burgers' equations in which all four mechanisms – linear evolution, geometrical spreading, convective nonlinearity and viscous diffusion – are of comparable importance. We shall term this region the 'nonlinear region'. Unfortunately, the only Burgers'-type equation for which a method of finding a solution to an initial value (or equivalently a boundary value) problem exists, is the ordinary Burgers' equation itself (Nimmo & Crighton 1982), where such a solution is furnished by the Cole–Hopf transformation (Hopf 1950; Cole 1951). Furthermore, apart from a similarity solution for the cylindrical Burgers' equation ((2.3) with $A(R) = R$)), (Chong & Sirovich 1973; Rudenko & Soluyan 1977; Scott 1981 *b*), which is not appropriate to either of the boundary conditions we are considering, there are no known non-trivial solutions to any other generalized Burgers' equations. Therefore, only when the motion in the nonlinear region is governed by the ordinary Burgers' equation may we find an exact solution to the problem. Despite this, if the non-uniformity is localized, and the matching conditions are of a particularly simple form, then we will be able, by using a slight generalization of the technique used by Crighton & Scott (1979) for N-waves with spherical spreading, to find the asymptotics of the solution within the nonlinear region. The result of this is that we may then *precisely determine the long-range* (*or old-age*) *evolution*. In the remaining cases we shall be unable to find the long-range solution completely, but will obtain it up to an undetermined constant. We believe that it is not possible to find the value of this constant by asymptotic matching arguments, despite the work of Enflo (1981); the reasons why Enflo's approach must fail were given in the Introduction.

First, however, we shall find the scalings for the nonlinear region for different classes of area functions. Again we consider harmonic and N-waves separately.

(*a*) *Harmonic waves*

Where possible we shall use the 'natural' scalings for R, θ and U. We scale R by the range $R_1(\epsilon)$ at which the non-uniformity comes about; θ is scaled by the 'thickness' of the region at that range, i.e. the shock thickness in the case of a localized non-uniformity and the scale associated with the whole wave for a gross non-uniformity. The dependent variable U will be scaled by the amplitude of the lossless solution at the range $R_1(\epsilon)$; this amplitude is $O(A^{-\frac{1}{2}}(R_1)\, I_A^{-1}(R_1))$. The natural scaling for R fails to be effective when the area changes are of an exponential or 'pseudo-exponential' nature, and the need to balance this behaviour with the algebraic behaviour elsewhere in the equation means that a combination 'stretch–shift', rather than the usual 'stretch', is needed. This state of affairs was also encountered, for different reasons, in the case of the spherical Burgers' equation by Crighton & Scott (1979).

If $A \in \mathrm{A}_{\mathrm{e}^-}$ then no non-uniformity arises; if $A \in \mathrm{A}_{\mathrm{pe}^-}$ or $A \in \bigcup_{\lambda<2} \mathrm{A}_\lambda$ then the non-uniformity is gross; and if $A \in \mathrm{A}_2^D$, $\mathrm{A} \in \mathrm{A}_2^C \bigcup_{\lambda>2} \mathrm{A}_\lambda$ or $A \in \mathrm{A}_{\mathrm{e}^+}$ then the non-uniformity is localized. We consider the five cases where non-uniformity arises, separately.

(i) $A \in \mathrm{A}_{\mathrm{pe}^-}$

This is a case in which A is like an exponential function, and so we take

$$\tilde{R} = \epsilon(R - R_1), \quad \tilde{\theta} = \theta, \quad U = \epsilon(\tilde{U}_0 + o(1)), \tag{5.1}$$

where $1 = \epsilon A^{\frac{1}{2}}(R_1)\, I_A(R_1) = O[\epsilon A(R_1)/A'(R_1)]$, and then we obtain

$$\tilde{U}_{0\tilde{R}} - z_0\, \tilde{U}_0\, \tilde{U}_{0\tilde{\theta}} - \tfrac{1}{2}\tilde{U}_0 = z_0\, \tilde{U}_{0\,\tilde{\theta}\,\tilde{\theta}}. \tag{5.2}$$

This is the Burgers' equation generalized to take into account area changes of the form $\tilde{A} = \mathrm{e}^{-\tilde{R}}$.

(ii) $A \in \bigcup_{\lambda<2} \mathrm{A}_\lambda$

By using the fact that $R_1 = \epsilon^{-1}$ in conjunction with the asymptotic form of I_A contained in Appendix II, we set

$$\tilde{R} = \epsilon R, \quad \tilde{\theta} = \theta, \quad U = \epsilon[\tilde{U}_0 + o(1)], \tag{5.3}$$

giving

$$\tilde{U}_{0\,\tilde{R}} - z_0\, \tilde{U}_0\, \tilde{U}_{0\,\tilde{\theta}} + (\lambda/2\tilde{R})\, \tilde{U}_0 = z_0\, \tilde{U}_{0\,\tilde{\theta}\tilde{\theta}}, \quad \lambda < 2, \tag{5.4}$$

the generalized Burgers' equation with $\tilde{A} = \tilde{R}^\lambda, \lambda < 2$.

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(iii) $A \in \mathrm{A}_2^D$

Here we have $\epsilon R_1(\ln R_1)^2 = 1$ and we make the scalings

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \theta/(\epsilon R_1)^{\frac{1}{2}}, \quad U = (R_1 \ln R_1)^{-1}(\tilde{U}_0 + o(1)), \tag{5.5}$$

which give

$$\tilde{U}_{0\tilde{R}} - z_0 \tilde{U}_0 \tilde{U}_{0\tilde{\theta}} + (1/\tilde{R})\tilde{U}_0 = z_0 \tilde{U}_{0\tilde{\theta}\tilde{\theta}}, \tag{5.6}$$

the spherical Burgers' equation ($\tilde{A} = \tilde{R}^2$).

(iv) $A \in \mathrm{A}_2^C \cup_{\lambda>2} \mathrm{A}_\lambda$

In this case $\epsilon A'(R_1) = 1$ and we take

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \theta/(\epsilon R_1)^{\frac{1}{2}}, \quad U = (\epsilon/R_1)^{\frac{1}{2}}[\tilde{U}_0 + o(1)], \tag{5.7}$$

leading to

$$\tilde{U}_{0\tilde{R}} - z_0 \tilde{U}_0 \tilde{U}_{0\tilde{\theta}} + (\lambda/2\tilde{R})\tilde{U}_0 = z_0 \tilde{U}_{0\tilde{\theta}\tilde{\theta}}, \quad \lambda \geqslant 2, \tag{5.8}$$

corresponding to $\tilde{A} = \tilde{R}^\lambda, \lambda \geqslant 2$.

(v) $A \in \mathrm{A}_{\mathrm{e}^+}$

Once more we have $\epsilon A'(R_1) = 1$, and because A behaves exponentially, we take the combination stretch–shift scaling for R defined by

$$\tilde{R} = [A'(R_1)/A(R_1)](R - R_1), \quad \tilde{\theta} = \theta/\epsilon A^{\frac{1}{2}}(R_1), \quad U = A^{-\frac{1}{2}}(R_1)[\tilde{U}_0 + o(1)], \tag{5.9}$$

and thus obtain

$$\tilde{U}_{0\tilde{R}} - z_0 \tilde{U}_0 \tilde{U}_{0\tilde{\theta}} + \tfrac{1}{2}\tilde{U}_0 = z_0 \tilde{U}_{0\tilde{\theta}\tilde{\theta}}, \tag{5.10}$$

the generalized Burgers' equation with area function $\tilde{A} = \mathrm{e}^{\tilde{R}}$.

For N-waves the situation is quite similar.

(b) *N-waves*

In this case a non-uniformity arises provided $A \in \mathrm{A}^+$. If $A \in \mathrm{A}_0^+$ then the non-uniformity is translational, if $A \in \cup_{0<\lambda<2} \mathrm{A}_\lambda$ the non-uniformity is gross; and if $A \in \mathrm{A}_2^D, A \in \mathrm{A}_2^C \cup_{\lambda>2} \mathrm{A}_\lambda$ or $A \in \mathrm{A}_{\mathrm{e}^+}$, the non-uniformity will be localized. The scalings are again chosen in the natural way when this is possible. As for harmonic waves, the natural scaling for the range variable is not successful for exponential-like area functions. Also, when the non-uniformity is translational, we would not expect the measures of shock width and height to be appropriate since the shock location is no longer known. In this case we use the natural scaling for R and the 'distinguished scalings' (Cole 1968) for θ and U.

(i) $A \in \mathrm{A}_0^+$

The required (distinguished) scaling is, with $\epsilon A^{\frac{1}{2}}(R_1) \ln R_1 = 1$,

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \theta/(\epsilon R_1)^{\frac{1}{2}}, \quad U = (\epsilon/R_1)^{\frac{1}{2}}(\tilde{U}_0 + o(1)), \tag{5.11}$$

and this gives the plane Burgers' equation, (5.4) with $\lambda = 0$.

(ii) $A \in \cup_{0<\lambda<2} \mathrm{A}_\lambda$

In this case $R_1(\epsilon)$ satisfies $\epsilon A^{\frac{1}{2}}(R_1) = 1$, and we take

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \theta/I_A^{\frac{1}{2}}(R_1), \quad U = A^{-\frac{1}{2}}(R_1)\, I_A^{-\frac{1}{2}}(R_1)\,(\tilde{U}_0 + o(1)), \tag{5.12}$$

giving (5.4) with $0 < \lambda < 2$.

(iii) $A \in \mathrm{A}_2^D$

Here $\epsilon A^{\frac{1}{2}}(R_1) \ln R_1 = 1$ and we take

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \frac{\theta - [1+z_\infty I_A(R)]^{\frac{1}{2}}}{\epsilon A^{\frac{1}{2}}(R_1) I_A^{\frac{1}{2}}(R_1)}, \quad U = A^{-\frac{1}{2}}(R_1)\, I_A^{-\frac{1}{2}}(R_1)\, [\tilde{U}_0 + o(1)], \tag{5.13}$$

giving (5.6).

(iv) $A \in \mathrm{A}_2^C \cup_{\lambda>2} \mathrm{A}_\lambda$

We have $\epsilon A'(R_1) = 1$ and appropriate scalings are

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \frac{\theta - [1+z_\infty I_A(R)]^{\frac{1}{2}}}{(\epsilon R_1)^{\frac{1}{2}}}, \quad U = \left(\frac{\epsilon}{R_1}\right)^{\frac{1}{2}} [\tilde{U}_0 + o(1)], \tag{5.14}$$

giving (5.8).

Finally,

(v) $A \in \mathrm{A}_{\mathrm{e}^+}$

Again $\epsilon A'(R_1) = 1$ and now the appropriate scalings are

$$\tilde{R} = R/R_1, \quad \tilde{\theta} = \frac{\theta - [1+z_\infty I_A(R)]^{\frac{1}{2}}}{\epsilon A^{\frac{1}{2}}(R_1)}, \quad U = A^{-\frac{1}{2}}(R_1)\, (\tilde{U}_0 + o(1)), \tag{5.15}$$

giving (5.10).

We have therefore shown that the motion in the nonlinear region is always governed by generalized Burgers' equations with one of the area functions $\tilde{A}(\tilde{R}) = \mathrm{e}^{-\tilde{R}}, \tilde{R}^\lambda, \mathrm{e}^{\tilde{R}}$ (for some $\lambda \in \mathbb{R}$). In the following two sections we shall discuss the solutions to these equations and the behaviour of the wave at long range.

6. Solution following a localized non-uniformity

In §5 we showed that the flow in the nonlinear region is governed by the generalized Burgers' equation

$$\tilde{U}_{0\,\tilde{R}} - z_0\, \tilde{U}_0\, \tilde{U}_{0\,\tilde{\theta}} + \tfrac{1}{2}(\tilde{A}'/\tilde{A})\,(\tilde{R})\,\tilde{U}_0 = z_0\, \tilde{U}_{0\,\tilde{\theta}\tilde{\theta}}, \tag{6.1}$$

where, following a localized non-uniformity, we have $\tilde{A} = \tilde{R}^\lambda, \lambda \geqslant 2$, or $\tilde{A} = \mathrm{e}^{\tilde{R}}$. Now, if we scale the geometrical effects out of $\tilde{U}_0$ by taking $\tilde{W} = \tilde{A}^{\frac{1}{2}}(\tilde{R})\,\tilde{U}_0$, we find that $\tilde{W}$ must satisfy

$$\tilde{W}_{\tilde{R}} - z_0\, \tilde{A}^{-\frac{1}{2}}(\tilde{R})\, \tilde{W}\tilde{W}_{\tilde{\theta}} = z_0\, \tilde{W}_{\tilde{\theta}\tilde{\theta}}, \tag{6.2}$$

and for the remainder of this section we shall use this form of the equation.

We want a solution of (6.2) that satisfies matching conditions to the Taylor shock as $\tilde{R} \to 0$, and to the lossless solution as $\tilde{\theta} \to \pm\infty$ *for all* $\tilde{R}$, since the lossless solution is *still* valid outside the thin nonlinear region. If the latter condition takes the form

$$\tilde{W} \to C^\pm \text{ (constants) as } \tilde{\theta} \to \pm\infty, \tag{6.3}$$

then we can use the method in C.S. to determine the asymptotic form of $\tilde{W}$ as $\tilde{R} \to \infty$, namely an error-function solution of the linearized version of (6.2). Fortunately, we find that this is indeed the case for those area functions which give rise to localized non-uniformities in the order one composite; more precisely, it occurs for $A \in \cup_{\lambda \geqslant 2} \mathrm{A}_\lambda$ and $A \in \mathrm{A}_{e^+}$.

For $A \in \cup_{\lambda \geqslant 2} \mathrm{A}_\lambda \cup \mathrm{A}_{e^+}$, we are able, by using (6.2) with (6.3), to show that

$$\tilde{W} \sim \tilde{W}_0 = \tfrac{1}{2}[(C^+ + C^-) + (C^+ - C^-) \operatorname{erf}(\tilde{\theta}/(4z_0 \tilde{R})^{\frac{1}{2}})] \quad \text{as} \quad \tilde{R} \to \infty, \tag{6.4}$$

where values for $C^\pm$ will be given below, this being a solution of the *linearized* Burgers' equation,

$$\tilde{W}_{0\,\tilde{R}} = z_0 \tilde{W}_{0\,\tilde{\theta}\tilde{\theta}}. \tag{6.5}$$

We shall now show, for harmonic and N-waves, that (6.1) is only satisfied at range $R = O(R_1)$; at any greater range the motion is governed by the linearized form (6.5). It is then straightforward to find a solution to the linear equation which matches the asymptotics (6.4) of the 'evolutionary' shock solution $\tilde{U}_0$ and the lossless solution, and from there to examine the transition to the 'long-range' régime. The long-range solutions we shall obtain are analogous to the old-age solutions obtained in initial value problems, and are themselves commonly referred to as 'old-age' solutions.

(*a*) *Harmonic waves*

(i) $A \in \cup_{\lambda \geqslant 2} \mathrm{A}_\lambda$

The thickness of the evolutionary shock given by (6.4) is $O((\epsilon R)^{\frac{1}{2}})$ and the amplitude of the lossless wave (in terms of W) is $O(I_A^{-1}(R))$. Therefore, if we let $R_2(\epsilon)$ be any function of ϵ such that $R_1 = o(R_2)$, the appropriate scalings at range $R = O(R_2(\epsilon))$ will be

$$\bar{R} = R/R_2, \quad \bar{\theta} = \theta/(\epsilon R_2)^{\frac{1}{2}}, \quad W = I_A^{-1}(R_2)\,(\overline{W}_0 + o(1))$$

and so, from (2.5), we obtain

$$\overline{W}_{0\,\bar{R}} = z_0 \overline{W}_{0\,\bar{\theta}\bar{\theta}}; \tag{6.6}$$

that is, at any range greater than $O(R_1)$ the shock is governed by (6.6). Not surprisingly, the solution to (6.6) which satisfies the matching requirements is

$$\overline{W}_0 = (p_\infty/z_\infty) \operatorname{erf}[\bar{\theta}/(4z_0 \bar{R})^{\frac{1}{2}})], \tag{6.7}$$

where $p_\infty = \lim_{R\to\infty} p_0$ and $p_0 = z_\infty I_A(R) \sin p_0$ (see (3.1)); for $A \in \mathrm{A}_2^D, p_\infty = \pi$ and $z_\infty = z_0$.

At a further range $R = O[R_3(\epsilon)]$, where $\epsilon R_3 = 1$, a *final* non-uniformity arises because the new error function shock thickness has become $O(1)$, and so this time the non-uniformity is gross. The natural scaling for this long-range region is

$$\hat{R} = R/R_3, \quad \hat{\theta} = \theta, \quad W = I_A^{-1}(R_3)\,[\hat{W}_0 + o(1)], \tag{6.8}$$

and then $\hat{W}_0$ must satisfy

$$\hat{W}_{0\,\hat{R}} = z_0 \hat{W}_{0\,\hat{\theta}\hat{\theta}}, \tag{6.9}$$

across the whole wave now, rather than in a narrow shock region. We require an odd 2π-periodic solution to (6.8) which matches the lossless solution and the linear shock solution (6.7), and hence obtain

$$\hat{W}_0 = \sum_{n=1}^{\infty} \frac{2p_\infty}{n\pi z_\infty} \exp\{-n^2 z_0 \hat{R}\} \sin n\hat{\theta}. \tag{6.10}$$

For large R, the fundamental mode will dominate and so we find that the ultimate old-age or long-range solution is

$$u = \frac{2p_\infty u_0}{\pi z_\infty} A^{-\frac{1}{2}}\left(\frac{r}{r_0}\right) I_A^{-1}(\epsilon^{-1}) \exp\{-\alpha r\} \sin \omega\tau, \tag{6.11}$$

where $\alpha = \delta\omega^2/2a_0^3$ is the usual small-signal attenuation constant. If $A \in \mathrm{A}_2^D$, then

$$u = \frac{4a_0^2}{(\gamma+1)\,\omega r \ln \epsilon^{-1}}\left[B\left(\frac{r}{r_0}\right)/B(\epsilon^{-1})\right]^{-\frac{1}{2}} \exp\{-\alpha r\} \sin \omega\tau, \tag{6.12}$$

and if $A \in \mathrm{A}_2^C \cup_{\lambda>2} \mathrm{A}_\lambda$ then

$$u = \frac{4I_A(\infty)\,a_0^2 p_\infty}{\pi(\gamma+1)\,\omega r_0} A^{-\frac{1}{2}}\left(\frac{r}{r_0}\right) \exp\{-\alpha r\} \sin \omega\tau, \tag{6.13}$$

where $B(R) = A(R)/R^\lambda$ is such that $RB'/B \to 0$, that is, B has less than algebraic variation, and $I_A(\infty) = \lim_{R\to\infty} I_A(R)$. Compare (6.12) with the old-age solution for the spherical Burgers' equation in C.S. (p. 127, equation (5.5)).

(ii) $A \in \mathrm{A}_{e^+}$

Because here the area functions are exponential-like we must introduce an unusual stretch–shift in the range variable. Consider a new thin region at range $R = O(R_2(\epsilon))$, where R_2 is such that $R_1 = o(R_2)$, in which we define the range variable to be

$$\bar{R} = (A'/A)\,(R_2)\,(R-R_2). \tag{6.14}$$

At range R_2 the shock thickness is $O([\epsilon(A/A')\,(R_2)]^{\frac{1}{2}})$ (in terms of $\bar{R}$) and in terms of W the amplitude of the lossless solution at the shock is $O(1)$, so that we set

$$\bar{\theta} = \theta/[\epsilon(A/A')\,(R_2)]^{\frac{1}{2}}, \quad W = \bar{W}_0 + o(1), \tag{6.15}$$

with the result that $\bar{W}$ must satisfy (6.6), as expected. Once more the required solution of (6.6) is (6.7).

It is clear that a gross non-uniformity will arise if the evolutionary shock becomes of $O(1)$ thickness at some range $R = O[R_3(\epsilon)]$ and that this will occur provided there exists an $R_3(\epsilon)$ such that $\epsilon A(R_3)/A'(R_3) = 1$. This will be the case if $A'/A \to 0$.

When this happens, things are very similar to case (i) above; we set

$$\hat{R} = [A'(R_3)/A(R_3)]\,(R-R_3), \quad \hat{\theta} = \theta, \quad W = \hat{W}_0 + o(1), \tag{6.16}$$

and find that $\hat{W}_0$ must satisfy (6.9) with solution (6.10), which leads to the old-age solution (6.13).

If such a gross non-uniformity does not arise then we have $A'/A \to \beta \in (0, \infty]$ and *the only possible non-uniformity then occurs in the lossless solution*. Such a non-uniformity arises when $\epsilon(W_1/W_0) = O(1)$, i.e. at range $R = O(R_3(\epsilon))$, where $\epsilon R_3 = 1$. It is clear that we should again have $\hat{R}$, $\hat{\theta}$ and $\hat{W}_0$ given by (6.16), where $R_3 = \epsilon^{-1}$ now, and in this way we find that

$$\hat{W}_{0\,\hat{R}} = 0, \tag{6.17}$$

which implies that at long range the leading order behaviour of u is only influenced by geometrical effects. This, perhaps surprising, fact may also be observed by considering (2.1) with $A'/A \to \beta \neq 0$. We see that in this case it is not possible to balance the geometrical and dissipative terms across the whole wave; at long range geometry must dominate. The required solution to (6.17) is

$$\hat{W}_0 = (p_\infty/\pi z_\infty)[(2n+1)\pi-\hat{\theta}] \quad \text{for} \quad \hat{\theta}\in(2n\pi, 2(n+1)\pi], \tag{6.18}$$

a sawtooth of period 2π and amplitude fixed in order to match the lossless solution and the solution in the shock region (6.7). In this way we have obtained the old-age solution

$$u = \frac{2a_0^2 p_\infty I_A(\infty)}{(\gamma+1)\,\omega r_0\,\pi} A^{-\frac{1}{2}}\left(\frac{r}{r_0}\right)[(2n+1)\pi-\omega\tau], \quad \text{for} \quad \omega\tau\in(2n\pi, 2(n+1)\pi]. \tag{6.19}$$

At this range the shock solution is still given by (6.7). This case, which includes the exponential horn, exhibits rather more interesting behaviour than one might have expected; although the dynamics are of the old-age kind in the sense that the governing equations are linear everywhere, the wave does not assume the fundamental sinusoidal wave form which ultimately emerges whenever the horn diverges less rapidly than exponentially and as represented typically by spherical waves.

(b) *N-waves*

The situation here is very similar to that describing harmonic waves above and so we omit many details.

(i) $A\in\bigcup_{\lambda\geqslant 2}\mathrm{A}_\lambda$

For any range $R_2(\epsilon)$ greater than $O(R_1)$ we have

$$\overline{W}_0 = -\tfrac{1}{2}C\,\mathrm{erfc}\,[\overline{\theta}/(4z_0\overline{R})^{\frac{1}{2}}], \tag{6.20}$$

where
$$\overline{R} = R/R_2, \quad \overline{\theta} = \frac{\theta-[1+z_\infty I_A(R)]^{\frac{1}{2}}}{(\epsilon R_2)^{\frac{1}{2}}},$$

$W = I_A^{\frac{1}{2}}(R_2)\,(\overline{W}_0+o(1))$ and $C = 1$ for $A\in\mathrm{A}_2^D$ and $C = (1+z_\infty)^{-\frac{1}{2}}$ otherwise. A gross non-uniformity comes about at range $R = O(R_3(\epsilon))$, where $\epsilon R_3 I_A^{-1}(R_3) = 1$, and so we introduce the scaling

$$\hat{R} = R/R_3, \quad \hat{\theta} = \theta I_A^{-\frac{1}{2}}(R_3), \quad W = I_A^{\frac{1}{2}}(R_3)\,[\hat{W}_0+o(1)],$$

which implies that $\hat{W}_0$ must satisfy (6.9) with solution

$$\hat{W}_0 = -\tfrac{1}{2}C\left[\hat{\theta}\left(\mathrm{erfc}\left(\frac{\hat{\theta}-1}{(4z_0\hat{R})^{\frac{1}{2}}}\right)-\mathrm{erfc}\left(\frac{\hat{\theta}+1}{(4z_0\hat{R})^{\frac{1}{2}}}\right)\right)-\frac{(4z_0\hat{R})^{\frac{1}{2}}}{\sqrt{\pi}}\left(\exp\left(-\frac{(\hat{\theta}-1)^2}{4z_0\hat{R}}\right)-\exp\left(-\frac{(\hat{\theta}+1)^2}{4z_0\hat{R}}\right)\right)\right], \tag{6.21}$$

which one may obtain by taking the matching condition as $\hat{R}\to 0$ to the lossless solution as a boundary condition at $\hat{R} = 0$ for the linear diffusion equation (6.9). The solution (6.21) also matches the evolutionary shock solution (6.20). We then find that, as $\hat{R}\to\infty$, (6.21) tends to a dipole solution of (6.9) which, in original variables, is

$$u = -\frac{4Cu_0 I_A^{\frac{1}{2}}(R_3)\,\omega\tau}{3\sqrt{\pi}\,(4\alpha r)^{\frac{3}{2}}} A^{-\frac{1}{2}}\left(\frac{r}{r_0}\right)\exp\left\{-\frac{\omega^2\tau^2}{4\alpha r}\right\}. \tag{6.22}$$

(ii) $A \in \mathrm{A}_{e^+}$

The solution in the shock region for any range $R_2(\epsilon)$ greater than $O(R_1)$ is

$$\overline{W}_0 = -\tfrac{1}{2}(1+z_\infty)^{-\frac{1}{2}} \operatorname{erfc}\,(\overline{\theta}/(4z_0\,\overline{R})^{\frac{1}{2}}), \tag{6.23}$$

where
$$\overline{R} = \frac{A'}{A}(R_2)\,(R-R_2), \quad \overline{\theta} = \frac{\theta-[1+z_\infty\,I_A(R)]^{\frac{1}{2}}}{[\epsilon(A/A')\,(R_2)]^{\frac{1}{2}}}$$

and $W = \overline{W}_0 + o(1)$. As for harmonic waves, this evolutionary shock will eventually thicken, if the area function is such that $A'/A \to 0$, or remain thin otherwise. In the former case a gross non-uniformity arises at range $R = O(R_3(\epsilon))$, where $\epsilon A(R_3)/A'(R_3) = 1$ and hence we make the scalings

$$\hat{R} = [A'(R_3)/A(R_3)]\,(R-R_3), \quad \hat{\theta} = \theta \quad \text{and} \quad W = \hat{W}_0 + o(1), \tag{6.24}$$

whence we find that $\hat{W}_0$ satisfies (6.9), but this time that linear equation governs motion across the whole wave. If the evolutionary shock remains thin then we expect no further non-uniformity since in the N-wave case the lossless solution remains uniform to arbitrarily large ranges and so the old-age motion is described by lossless arcs linked by *linear* shocks of constant thickness, given by (6.23).

This completes the study of the solution following a localized breakdown in the order one composite.

7. Solution following a gross or translational non-uniformity

When a breakdown in the order one composite of one of these kinds occurs, we seek solutions of generalized Burgers' equation (6.1) which match the Taylor-shock and lossless solutions. For harmonic waves, $\tilde{A} = e^{-\tilde{R}}$ corresponding to $A \in \mathrm{A}_{pe^-}$; the pseudo-exponential area functions, or $\tilde{A} = \tilde{R}^\lambda$ corresponding to $A \in \mathrm{A}_\lambda$ for $\lambda < 2$, and for N-waves $\tilde{A} = 1$ for $A \in \mathrm{A}_0^+$ or $\tilde{A} = \tilde{R}^\lambda$, corresponding to $A \in \mathrm{A}_\lambda$ for $0 < \lambda < 2$. As discussed in §5, we can only construct such solutions when $\tilde{A} = 1$ and otherwise the most we can do is to find the form of the long-range solution. However, by going through the matching procedure up to this stage we have reduced the original problem, which depended on the six parameters δ, ω, a_0, u_0, r_0 and γ together with the area function A, to a set of canonical problems depending on only one dimensionless parameter z_0, as given in (2.1), and the *generic* area functions $\tilde{A}$.

For all the area functions under discussion in this section it may be seen that the amplitude of the wave given by the lossless solution, which is $O(A^{-\frac{1}{2}}(R)\,I_A^{-1}(R))$ $O(A^{-\frac{1}{2}}(R)\,I_A^{-\frac{1}{2}}(R))$ in the harmonic (N-wave) case, decreases for large R up to the range of the breakdown. One would not expect this state of affairs to change after the breakdown and so we assume that in each case the amplitude of the wave will decrease so that, at large $\tilde{R}$, the nonlinear term in (6.1) will become sufficiently small that it may be ignored and hence the long-range solution will be given by suitable solutions of the linearized form

$$\tilde{U}_{0\,\tilde{R}} + \tfrac{1}{2}(\tilde{A}'/\tilde{A})\,(\tilde{R})\,\tilde{U}_0 = z_0\,\tilde{U}_{0\,\tilde{\theta}\tilde{\theta}}. \tag{7.1}$$

For harmonic waves we seek odd 2π-periodic solutions to (7.1) and thus as $\tilde{R} \to \infty$ we find the 'old-age' form

$$\tilde{U}_0 \sim C(z_0;\tilde{A})\,\tilde{A}^{-\frac{1}{2}}(\tilde{R})\,\exp\{-z_0\,\tilde{R}\}\sin\tilde{\theta}, \tag{7.2}$$

and consequently, in dimensional variables we obtain

$$u \sim C(z_0; \tilde{A})\, u_0 \epsilon \exp\{(z_0 - \tfrac{1}{2})\, \epsilon R_1(\epsilon)\} \exp\{(\epsilon/2r_0)\, r\} \exp\{-\alpha r\} \sin \omega\tau, \tag{7.3}$$

for $A \in \mathrm{A}_{\mathrm{pe}^-}$, where $\epsilon A^{\frac{1}{2}}(R_1)\, I_A(R_1) = 1$, and

$$u \sim C(z_0; \tilde{A})\, u_0 \alpha^{-\frac{1}{2}\lambda} r^{-\frac{1}{2}\lambda} \exp\{-\alpha r\} \sin \omega\tau, \tag{7.4}$$

for $A \in \mathrm{A}_\lambda$, where $\lambda < 2 \neq 0$.

In a similar way we find for N-waves that

$$\tilde{U}_0 \sim D(z_0, \tilde{A})\, \tilde{R}^{-\frac{1}{2}(\lambda+3)} \tilde{\theta} \exp\{-\tilde{\theta}^2/4z_0 \tilde{R}\}, \tag{7.5}$$

for $A \in \mathrm{A}_\lambda$ with $0 < \lambda < 2$.

In those cases in which the generic area function $\tilde{A}$ is unity one is able to find the solution precisely. For harmonic waves this occurs for all area functions $A \in \mathrm{A}_0$ and, in the N-wave case, for those 'converging' area functions in A_0, namely $A \in \mathrm{A}_0^+$.

In each of these two cases we obtain the solution to (6.1) (with $\tilde{A} = 1$) by means of the Cole–Hopf transformation, taking the matching condition to the lossless solution as the 'initial' value; for harmonic waves we require that

$$\tilde{U}_0 \sim -[\tilde{\theta} - (2n+1)\,\pi]/z_0 \tilde{R} \quad \text{for} \quad \theta \in (2n\pi, 2(n+1)\,\pi] \quad \text{as} \quad \tilde{R} \to 0, \tag{7.6}$$

from which we obtain

$$\tilde{U}_0 = \frac{4 \sum_{n=1}^{\infty} n \exp\{-n^2 z_0 \tilde{R}\} \sin n\tilde{\theta}}{1 + 2 \sum_{n=1}^{\infty} \exp\{-n^2 z_0 \tilde{R}\} \cos n\tilde{\theta}}. \tag{7.7}$$

From this solution we obtain the long-range solution by letting $\tilde{R} \to \infty$, whence

$$\tilde{U}_0 \sim 4 \exp\{-z_0 \tilde{R}\} \sin \tilde{\theta}, \tag{7.8}$$

or in dimensional variables

$$u \sim \frac{4\omega\delta}{a_0(\gamma+1)} \exp\{-\alpha r\} \sin \omega\tau, \tag{7.9}$$

in agreement with the well-known result (see, for example, Shooter *et al.* 1974). It may also be shown that (7.7) also fulfils the second requirement, namely it matches the Taylor-shock solution in the limit $\tilde{R} \to 0, \tilde{\theta} \to 0$.

For N-waves we find that

$$\tilde{U}_0 = -\frac{\tilde{\theta}}{\tilde{R}} \left(1 + \tilde{R}^{\frac{1}{2}} \exp\left\{\frac{\tilde{\theta}^2}{4z_0 \tilde{R}}\right\}\right)^{-1} \tag{7.10}$$

in a similar way (see also C.S., p. 113, equation 3.28), and as $\tilde{R} \to \infty$, tends to a dipole solution of the linearized (7.1) (with $\tilde{A} = 1$),

$$\tilde{U}_0 \sim -(\tilde{\theta}/\tilde{R}^{\frac{3}{2}}) \exp\{-\tilde{\theta}^2/4z_0 \tilde{R}\}. \tag{7.11}$$

In the next section we shall give a more detailed discussion of the N-wave problem for the Burgers' equation with reference to the exact solution to (2.1 and 2.2).

This completes the discussion of sound propagation in a horn in the case of small diffusivity.

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In §9 we shall return to this problem to consider the question of amplitude saturation for a horn of arbitrary cross-sectional area.

8. Exact solution to the plane Burgers' equation with N-wave boundary condition

If the cross-sectional area of the horn is constant for all R then the flow is governed by the plane Burgers' equation ((2.3) with $A(R) = \text{const.}$) and, as discussed in §5, this equation is unique among the class of generalized Burgers' equations in that its initial or boundary value problems may be solved exactly by means of the famous Cole–Hopf transformation. In order to compare with the results obtained above using matched asymptotic expansions, and better to understand the nature of the 'translational' non-uniformity in the Taylor-shock solutions that occur for plane waves, we shall use this technique to obtain the exact solution to the N-wave problem discussed above.

In order to compare the exact solution with the lossless and Taylor-shock solutions, (3.7) and (AI 3, AI 4), we again transform from U and R to W and z and consider the problem

$$W_z - WW_\theta = \epsilon W_{\theta\theta} \tag{8.1}$$

with

$$W(0,\theta) = \begin{Bmatrix} -\theta & |\theta| < 1 \\ 0 & |\theta| > 1 \end{Bmatrix}, \tag{8.2}$$

where $z = z_0(R-1)$ and $W = U$. By means of the Cole–Hopf transformation,

$$W(z,\theta) = 2\epsilon(\partial/\partial\theta)\ln\psi(z,\theta), \tag{8.3}$$

where ψ satisfies the linear diffusion equation

$$\psi_z = \epsilon\psi_{\theta\theta} \tag{8.4}$$

subject to the boundary condition

$$\psi(0,\theta) = \begin{Bmatrix} \exp\{-(\theta^2-1)/4\epsilon\} \\ 1 \end{Bmatrix}, \tag{8.5}$$

where constants of integration in (8.5) are chosen in order to make $\psi(0,\theta)$ continuous. It is now straightforward to solve (8.4) with (8.5) to obtain

$$\psi(z,\theta) = 1 - \frac{1}{2}\left[\operatorname{erf}\left(\frac{1-\theta}{(4\epsilon z)^{\frac{1}{2}}}\right) + \operatorname{erf}\left(\frac{1+\theta}{(4\epsilon z)^{\frac{1}{2}}}\right)\right] + \tfrac{1}{2}(1+z)^{-\frac{1}{2}}\exp\left\{-\frac{\theta^2-(1+z)}{4\epsilon(1+z)}\right\}$$
$$\left[\operatorname{erf}\left(\frac{1+z}{4\epsilon z}\right)^{\frac{1}{2}}\left(1-\frac{\theta}{1+z}\right) + \operatorname{erf}\left(\left(\frac{1+z}{4\epsilon z}\right)^{\frac{1}{2}}\left(1+\frac{\theta}{1+z}\right)\right)\right] \tag{8.6}$$

and so, by means of (8.3), we obtain the required solution, $W(z,\theta)$.

It may be shown that, provided that $S \equiv [\epsilon(1+z)]/[\theta^2-(1+z)] = o(1)$, we have, for *all* z,

$$W(z,\theta) = \begin{Bmatrix} -\theta/(1+z) + \text{exponentially small terms (EST)} & |\theta| < (1+z)^{\frac{1}{2}} \\ \text{EST} & |\theta| > (1+z)^{\frac{1}{2}} \end{Bmatrix} \tag{8.7}$$

in agreement with (3.3). If $S = O(1)$ then $\theta^2/(1+z)$ must be close to unity and so (8.7) is valid provided θ lies outside the thin regions around $\pm(1+z)^{\frac{1}{2}}$, which are known as the shock regions. Now if we concentrate our attention on the shock at $\theta = +(1+z)^{\frac{1}{2}}$ and take $z = O(1)$, we see that in this region $\theta+(1+z)^{\frac{1}{2}} = O(1)$ and hence that $\theta-(1+z)^{\frac{1}{2}} = O(\epsilon)$. In order to study the flow within the shock region, we therefore define the stretched variable $\theta^* = [\theta-(1+z)^{\frac{1}{2}}]/\epsilon$ and rewrite (8.3 and 8.6) in terms of it, to obtain

$$W(z,\theta) = -\tfrac{1}{2}(1+z)^{-\frac{1}{2}}[1+8\epsilon\chi-2\epsilon\ln(1+z)]^{\frac{1}{2}}(1-\tanh\chi)+\text{EST}, \tag{8.8}$$

where
$$\chi = \frac{\theta^*+(1+z)^{\frac{1}{2}}\ln(1+z)}{4(1+z)^{\frac{1}{2}}}+\frac{\epsilon\theta^{*2}}{8(1+z)}. \tag{8.9}$$

Expanding further in ϵ, we find that (8.8) gives the Taylor shock solution and its $O(\epsilon)$ correction (AI 3, AI 4). Note also that if we were to rewrite the two-term Taylor shock solution by including the function $l(z)$ in the argument of the hyperbolic tangent as a further correction to the shock location (as discussed in Appendix I), we would obtain (8.8). Therefore we see that (8.8) tells us that the 'shock centre location', up to exponentially small terms, is determined by setting the argument χ of the 'tanh' in (8.8) equal to zero, and is

$$\theta = (1+z)^{\frac{1}{2}}[1-2\epsilon\ln(1+z)]^{\frac{1}{2}}. \tag{8.10}$$

It is only possible to have $\chi = 0$ while $z \leqslant \exp(1/2\epsilon)-1$ and beyond this range the argument of the 'tanh' may take only positive values. Therefore it is clear that at range $z = O(\exp(1/2\epsilon))$ it becomes necessary to find a new description for the shock region since it is no longer possible to describe the transition from $W = 0$ to $W = (1+z)^{\frac{1}{2}}$ by means of a hyperbolic tangent. This range essentially corresponds to the range $R = O(e^{1/\epsilon})$ at which the breakdown in the order one composite is found to come about in the analysis of §4.

From the above we see that the term 'translational' is perhaps a misnomer; the shock has not moved to a distant unknown location, rather it is no longer possible to find a location to fit a 'tanh'-type shock. Remembering that there still exist *narrow regions* (with respect to the scale of the whole wave) around $\pm(1+z)^{\frac{1}{2}}$ in which some sort of shock exists, we now want a scaling for θ which exhibits this feature.

In order to achieve this, as for $z = O(1)$, we must scale θ in such a way that $(1+z)^{-\frac{1}{2}}\exp\{-[\theta^2-(1+z)]/4\epsilon(1+z)\}$ remains $O(1)$. Here, however, we have $z = O(\exp(1/2\epsilon))$ and so if we set $z_1(\epsilon) = \exp(1/2\epsilon)$ and $\tilde{z} = z/z_1$ we find that the exponential term is

$$\tilde{z}^{-\frac{1}{2}}\exp\{-\theta^2/4\epsilon z_1\tilde{z}\}$$

and hence we must take $\tilde{\theta} = \theta/(\epsilon z_1)^{\frac{1}{2}}$. Furthermore, by considering (8.3 and 8.6) we find that $W(z,\theta) = O(\epsilon/z_1)^{\frac{1}{2}}$ and so we have shown how the distinguished scalings given in (5.11) arise in a fairly natural way from consideration of the exact solution. From this scaling we may again obtain the long-range solution (7.11).

9. Amplitude saturation in a horn of arbitrary cross-sectional area

Amplitude saturation is the familiar property of nonlinear, rather than linear, acoustics in which, as the source amplitude is increased, the amplitude of the output is observed to approach a value which is independent of the source amplitude. In theoretical studies it is said that amplitude saturation occurs when the long-range (or old-age) solution is found not to depend on the amplitude of the source. Values of this saturation amplitude have been found for a time-harmonic source in the cases of plane and spherical symmetry (see Shooter *et al.* (1974) and references therein).

In the case of a horn, with sound propagation governed by (1.1), we must, in order to study this phenomenon, let u_0 tend to infinity while holding all other parameters fixed. Consequently, we must consider (2.1) in the limit as ϵ tends to zero with $\epsilon z_0 = \alpha r_0$, a fixed $O(1)$ constant. It is therefore relevant to consider (2.10 and 2.11) in the limit as $\epsilon \to 0$ with harmonic boundary condition, this being the one of principal interest in experiments where amplitude saturation might be expected.

We now obtain asymptotic expansions for $a(\epsilon, z)$ for small ϵ. If we let $F_A(R) = \int^R A^{-\frac{1}{2}}(R')\,dR'$ $(= (1/z_0)\, I_A(R))$, then, on inverting the transformation (2.11), we find that

$$R = F_A^{-1}(F_A(1) + \epsilon z/\alpha r_0)$$

and hence
$$R = 1 + \epsilon z/\alpha r_0 + O[(\epsilon z)^2], \tag{9.1}$$

giving
$$a(\epsilon, z) = \epsilon A^{\frac{1}{2}}(R) = \epsilon(1 + \tfrac{1}{2}A'(1)\,(\epsilon z/\alpha r_0) + O[(\epsilon z)^2]), \tag{9.2}$$

because we fixed $A(1) = 1$. Thus, provided that $z = o(\epsilon^{-1})$, the flow is governed by an equation which to leading order (in $a(\epsilon, z)$) is the *plane* Burgers' equation,

$$W_z - WW_\theta = \epsilon(1 + O(\epsilon z))\, W_{\theta\theta}. \tag{9.3}$$

By using the analysis of §§3, 4 (and Appendix I) we find that for $z = O(1)$ the solution may be described in terms of a lossless solution plus Taylor-shock solutions and that a gross non-uniformity arises when $z = O(\epsilon^{-1})$. When this occurs we must redefine the solution across the whole wave. The appropriate scalings at this range are

$$\tilde{z} = \epsilon z, \quad \tilde{\theta} = \theta, \quad W = \epsilon \tilde{W}_0 + o(\epsilon),$$

which give
$$\tilde{W}_{0\tilde{z}} - \tilde{W}_0\, \tilde{W}_{0\,\tilde{\theta}} = \tilde{a}(\tilde{z}, \alpha r_0)\, \tilde{W}_{0\,\tilde{\theta}\tilde{\theta}}, \tag{9.4}$$

where $\tilde{a}(\tilde{z}, \alpha r_0) = A^{\frac{1}{2}}(R)$ – the full generalized Burgers' equation to which, in general, an appropriate solution is not known. However, since (9.4) and the matching conditions to the lossless solution and Taylor-shock solutions are independent of ϵ (and hence u_0), we have been able to show that amplitude saturation occurs for all area functions $A(R)$. In fact all knowledge of the initial amplitude is lost before the range $z = O(\epsilon^{-1})$, or, in terms of the original variable, before $R = O(1)$, and hence clearly the long-range solutions must be independent of ϵ.

If we wish to investigate further the solutions to (9.4) it is most convenient to transform back to the original variables R and U. At the range of the non-uniformity $R = O(1)$ and $U = O(\epsilon)$ so that we must introduce the scalings

$$\tilde{R} = R, \quad \tilde{\theta} = \theta, \quad U = \epsilon[\tilde{U}_0 + o(1)]$$

which leads to
$$\tilde{U}_{0\tilde{R}} - \alpha r_0\, \tilde{U}_0\, \tilde{U}_{0\tilde{\theta}} + \tfrac{1}{2}(A'/A)\,(\tilde{R})\, \tilde{U}_0 = \alpha r_0\, \tilde{U}_{0\,\tilde{\theta}\tilde{\theta}}. \tag{9.5}$$

As discussed at the end of §7, we may approximate the solutions of (9.5) by solutions of the linearized equation

$$\tilde{U}_{0\tilde{R}} + \tfrac{1}{2}(A'/A)\,(\tilde{R})\, \tilde{U}_0 = \alpha r_0\, \tilde{U}_{0\,\tilde{\theta}\tilde{\theta}}, \tag{9.6}$$

provided that the area function is such that the amplitude of the waves decreases with range

$\tilde{R}$ so that the nonlinear term becomes insignificant. In the light of what was said in §7 we feel that this will be the case for all area functions that are not exponentially converging.

In these cases we obtain the old-age or long-range form as $\tilde{R} \to \infty$

$$u = C_A(\alpha r_0)[a_0^2/(\gamma+1)\,\omega r_0]\,A^{-\frac{1}{2}}(r/r_0)\,\exp\{-\alpha r\}\sin\omega\tau, \tag{9.7}$$

which exhibits the amplitude saturation that we showed had come about earlier.

In the same way that the limit considered here $(\epsilon \to 0, z_0 = O(\epsilon^{-1}))$ corresponds to an experiment in which the source amplitude is increased while other parameters are fixed, the limit considered in §§3–8 may also correspond to an experiment. If we let $f_0 = u_0\omega$ be the acceleration of the source, we have $\epsilon = \delta\omega^2/a_0 f_0(\gamma+1)$ and $z_0 = (\gamma+1)\,r_0 f_0/2a_0^2$. Now if we perform an experiment in which ω is decreased and f_0 and the remaining parameters are held fixed, then this experiment corresponds to $\epsilon \to 0, z_0 = O(1)$. Note that as we decrease ω we must *increase* u_0 in order that f_0 be fixed. However, for $A \in \cup_{\lambda \geqslant 2} \mathrm{A}_\lambda$ the amplitude of the long-range solution (see 6.12 and 6.13) is a *decreasing* function of u_0 and hence we have an unusual 'super-saturation' effect in which the output amplitude decreases as the source amplitude increases.

10. Comparisons with other approaches

In this section we discuss some approximate methods which have been previously proposed for dealing with the joint effects of geometrical spreading and attenuation on nonlinear acoustic propagation. We concern ourselves only with cases in which shocks have formed, and with the disintegration of the shock-wave pattern into old age at large ranges. For the early stages of the propagation, in which the cumulative effects of nonlinearity are not yet significant, it is straightforward to fully account for linear mechanisms such as geometrical spreading and attenuation while treating nonlinear mechanisms by a perturbation expansion in amplitude; this has been done by many authors for different circumstances and will not be further examined here. It should also be said that *many* approximate methods have been proposed for attacking the long-range problem. None of them, as far as we can see, are *rational* in the sense nowadays applied in singular perturbation theory, with the exception of the papers by Crighton & Scott (1979) and Scott (1981 *a*) which are forerunners of the present work. It is generally quite pointless therefore to try to relate their predictions to ours as it is not clear to what extent theirs should overlap with ours; the structure of their predictions is quite different from ours, although they may be numerically similar for limited parameter ranges. For further discussion and references the reader should consult Beyer (1984) and Webster (1977).

There are two approaches, however, which give predictions with quite intricate structure, and which differ from ours only in numerical multipliers, and we shall go into these in a little detail. Not all the references for these are easily accessible, and the notation is generally quite different, so that we shall carry out the work of this section in dimensional variables and give enough detail for the reader to make whatever comparisons seem appropriate. We deal exclusively with the sinusoidal boundary-value problem, and with the specific cases of plane, cylindrical and spherical waves. The model problems are thus defined by the equation

$$\frac{\partial u}{\partial r} - \frac{\gamma+1}{2a_0^2}\,u\frac{\partial u}{\partial \tau} + \frac{ju}{2r} + \alpha u = 0 \tag{10.1}$$

for the velocity $u(r,\tau)$, with

$$u = u_0 \sin\omega\tau \quad \text{at} \quad r = r_0.$$

Here r is the range variable, $\tau = t-(r-r_0)/a_0$ is retarded time and $j = 0, 1, 2$ for plane, cylindrical and spherical waves respectively. For the dissipative term, written as αu, we consider two cases: first, the thermoviscous case in which α is an operator, $\alpha u = -(\delta/2a_0^3)\,\partial^2 u/\partial\tau^2$ with δ the diffusivity of sound; and secondly, the constant attenuation case, in which α is a scalar constant. Such a term, αu, does indeed represent the attenuation of a relaxing medium for disturbances whose typical frequencies are above the relaxation frequency (Rudenko & Soluyan 1977, p. 85; Lighthill 1956, p. 276). More significantly, the choice of constant α is the only choice for which we can get exact solutions to (10.1) (except for $j = 0$) for comparison with approximate solutions. In the work below we give the exact solutions for plane, cylindrical and spherical waves, together with the exact solution of the Burgers' equation for plane thermoviscous motion; and we compare these with our own predictions, based on earlier sections above, and with the predictions associated with Rudnick (1953) and Shooter *et al.* (1974) (hereafter referred to as S.M.B.).

(*a*) *Plane waves*

Equation (10.1), with $j = 0$ and $x = r-r_0$, is reduced to plane wave form

$$\frac{\partial v}{\partial \xi} - \frac{\gamma+1}{2a_0^2}\, v\, \frac{\partial v}{\partial \tau} = 0 \tag{10.2}$$

with $v = \exp(\alpha x)\, u$ and $\xi = (1-e^{-\alpha x})/\alpha$. Thus, away from shocks (where detailed structure cannot be resolved by the constant α mechanism), we have

$$u = u_0\, e^{-\alpha x} \sin\omega\phi, \quad \phi-\tau = \tfrac{1}{2}[(\gamma+1)/a_0^2]\, u_0\, \xi \sin\omega\phi, \tag{10.3}$$

and as $\alpha x \to \infty$, ϕ becomes a function of τ alone (cf. the 'frozen' wave forms described in §6 above) with

$$\phi-\tau = \tfrac{1}{2}[(\gamma+1)/a_0^2]\, u_0\, \alpha^{-1} \sin\omega\phi.$$

For small attenuation, $\omega\phi$ must be near π (restricting attention to $0 < \omega\tau \leqslant 2\pi$) and we find

$$u \sim e^{-\alpha x}[2a_0^2\alpha/(\gamma+1)]\,(\pi/\omega-\tau) \tag{10.4}$$

in $0 < \omega\tau < 2\pi$, with periodic continuation elsewhere. Thus u has a periodic sawtooth form, with amplitude decay as $\exp(-\alpha x)$. Resolving u into its Fourier sine components,

$$u = \sum_{n=1}^{\infty} u_n(x) \sin n\omega\tau$$

we have
$$u_1(x) = [4a_0^2\alpha/(\gamma+1)\,\omega]\exp(-\alpha x) \tag{10.5}$$

in which this fundamental component (and of course the whole wave) has clearly suffered amplitude saturation.

The corresponding result for a thermoviscous medium, obtained from the Cole–Hopf linearization of Burgers' equation, is well known. With $\alpha = \delta\omega^2/2a_0^3$ here denoting the attenuation coefficient at the fundamental frequency ω, it is

$$u_1(x) = [8a_0^2\alpha/(\gamma+1)\,\omega]\exp(-\alpha x), \tag{10.6}$$

just twice the value in (10.5).

Now to apply Rudnick's (1953) method, we return to the lossless form of (10.3) by taking $\alpha = 0$ and $\xi = x$, so that the sawtooth of (10.4) becomes

$$u \sim \frac{2}{\gamma+1}\frac{a_0^2}{x}\left(\frac{\pi}{\omega}-\tau\right). \tag{10.7}$$

(This result is almost invariably quoted in the nonlinear acoustics literature at the next order of approximation as $x \to \infty$, in which case $(x+\bar{x})$ replaces x in the denominator, $\bar{x} = 2a_0^2/(\gamma+1)\,\omega u_0$ being the shock-formation range.) The fundamental Fourier coefficient associated with (10.7) is

$$u_1(x) = 4a_0^2/(\gamma+1)\,\omega x, \tag{10.8}$$

whose rate of change

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} = -\frac{4a_0^2}{(\gamma+1)\,\omega x^2} = -\frac{(\gamma+1)\,\omega u_1^2}{4a_0^2} \tag{10.9}$$

is entirely associated with the implicit energy dissipation within the shocks, dissipation outside the shocks so far being disregarded. This 'mainstream' dissipation, for which in the absence of nonlinear effects we would have

$$\mathrm{d}u_1/\mathrm{d}x = -\alpha u_1, \tag{10.10}$$

is now brought in by Rudnick's postulate that the total $\mathrm{d}u_1/\mathrm{d}x$ is that obtained by simply adding linear and nonlinear contributions, giving

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} + \alpha u_1 + \frac{(\gamma+1)\,\omega u_1^2}{4a_0^2} = 0 \tag{10.11}$$

and

$$u_1 = [C\,\mathrm{e}^{\alpha x} - \tfrac{1}{4}(\gamma+1)\,\omega/a_0^2\alpha]^{-1} \tag{10.12}$$

$$\sim [(C - \tfrac{1}{4}(\gamma+1)\,\omega/a_0^2\alpha) + (C\alpha)\,x + \ldots]^{-1} \tag{10.13}$$

as $\alpha x \to 0$. On the other hand, for $\bar{x} \ll x \ll \alpha^{-1}$, (10.7) should hold and mainstream dissipation should be unimportant, so that (10.13) and (10.7) should match. Thus

$$C = \tfrac{1}{4}(\gamma+1)\,\omega/a_0^2\alpha \tag{10.14}$$

from which it follows from (10.12) that for $\alpha x \gg 1$,

$$u_1 \sim [4a_0^2\alpha/(\gamma+1)\,\omega]\exp(-\alpha x), \tag{10.15}$$

and that for this particular case Rudnick's old-age formula and the exact result for constant α agree. Rudnick's method does not explicitly assume any particular variation, or lack of variation, of α with ω, and might therefore be expected to apply equally to thermoviscous attenuation. It does not, of course, giving an amplitude too small by a factor 2 (cf. (10.6)). This factor might possibly be 'explained' by the introduction of an 'effective' α in the thermoviscous case, but we shall see that Rudnick's method does not always work even in the case of constant α (although the structure of results obtained by this method seems always to be correct) so that it is not worth trying to force the equivalence further.

An alternative, related, proposal is made in S.M.B. One calculates the range $x_{\max}$ at which the logarithmic rates of change $\mathrm{d}(\ln u_1)/\mathrm{d}x$ associated with linear and nonlinear mechanisms are equal, obtaining from (10.8) and (10.10)

$$x_{\max} = \alpha^{-1}. \tag{10.16}$$

It is then asserted that $u_1(x)$ decays linearly for $x_{max} \leqslant x < \infty$ with $u_1(x_{max})$ given from (10.8). Thus

$$u_1(x) = [4a_0^2/(\gamma+1)\,\omega x_{max}]\exp\{-\alpha(x-x_{max})\} \tag{10.17}$$

$$\sim [4\mathrm{e}\,a_0^2\,\alpha/(\gamma+1)\,\omega]\exp(-\alpha x). \tag{10.18}$$

This has the same structure as (10.5) and (10.6) (and of course is independent of initial amplitude u_0), but the coefficients in (10.5), (10.6) and (10.18) are all different, 4, 8 and 4e, respectively. The results of our own asymptotic analysis in this paper, and previously in Crighton & Scott (1979), are identical with those of the Cole–Hopf solution (10.6) for the thermoviscous fluid, and with those of the exact solution leading to (10.5) in the constant α-case. Comparison of these plane-wave results with experiment will follow later.

(b) *Cylindrical waves*

Equation (10.1), with $j = 1$, is reduced to plane-wave form (10.2) with

$$v = r^{\frac{1}{2}}\mathrm{e}^{\alpha r}u$$

and

$$\xi = \int_{r_0}^{r}(\mathrm{e}^{-\alpha s}/s^{\frac{1}{2}})\,\mathrm{d}s = (1/\alpha^{\frac{1}{2}})\{E_{\frac{1}{2}}(\alpha r_0)-E_{\frac{1}{2}}(\alpha r)\}, \tag{10.19}$$

where

$$E_{\frac{1}{2}}(z) = \int_z^{\infty}(\mathrm{e}^{-t}/t^{\frac{1}{2}})\,\mathrm{d}t \tag{10.20}$$

is an exponential integral of order $\frac{1}{2}$. Assuming that shocks develop fully in the wave, the result corresponding to (10.4) is

$$u \sim \left(\frac{r_0}{r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha r}\frac{2a_0^2}{(\gamma+1)}\left(\frac{\alpha}{r_0}\right)^{\frac{1}{2}}\left(\frac{\pi}{\omega}-\tau\right)\frac{1}{E_{\frac{1}{2}}(\alpha r_0)}, \tag{10.21}$$

and corresponding to (10.5),

$$u_1 \sim \left(\frac{r_0}{r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha r}\left(\frac{a_0}{\gamma+1}\right)\left(\frac{a_0}{\omega r_0}\right)4\frac{(\alpha r_0)^{\frac{1}{2}}}{E_{\frac{1}{2}}(\alpha r_0)}. \tag{10.22}$$

It is interesting to look at the limits $\alpha r_0 \to 0, \alpha r_0 \to \infty$. For the former we find

$$u_1 \sim \left(\frac{1}{\pi\alpha r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha r}\left[\frac{4a_0^2\,\alpha}{(\gamma+1)\,\omega}\right], \tag{10.23}$$

with no dependence on r_0, with r scaled by the attenuation length α^{-1}, and where the quantity in square brackets is the plane-wave saturation amplitude of (10.5). When $\alpha r_0 \to \infty$, on the other hand,

$$u_1 \sim \left(\frac{r_0}{r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha(r-r_0)}\left[\frac{4a_0^2\,\alpha}{(\gamma+1)\,\omega}\right], \tag{10.24}$$

whose features are entirely satisfactory. The wave is effectively plane until well into old age, and has the plane-wave amplitude factor again; old age then continues from range r_0 under linear spreading and dissipative mechanisms.

To implement the Rudnick or S.M.B. proposals we have the lossless results

$$\left.\begin{aligned} \xi &= 2(r^{\frac{1}{2}}-r_0^{\frac{1}{2}}), \\ u &\sim \left(\frac{a_0^2}{\gamma+1}\right)\left(\frac{1}{r-(rr_0)^{\frac{1}{2}}}\right)\left(\frac{\pi}{\omega}-\tau\right), \\ u_1(r) &= \frac{2a_0^2}{(\gamma+1)\,\omega}\left(\frac{1}{r-(rr_0)^{\frac{1}{2}}}\right). \end{aligned}\right\} \tag{10.25}$$

Calculating $\mathrm{d}u_1/\mathrm{d}r$ from this (for $r \gg r_0$) and adding the linear contribution $-u_1/2r-\alpha u_1$ gives

$$\frac{\mathrm{d}u_1}{\mathrm{d}r}+\frac{u_1}{2r}+\alpha u_1+\left(\frac{2a_0^2}{(\gamma+1)\,\omega}\right)^{-1}u_1^2 = 0 \tag{10.26}$$

with solution

$$u_1 = \left[Cr^{\frac{1}{2}}\mathrm{e}^{\alpha r}-\frac{(\gamma+1)\,\omega}{2a_0^2}\left(\frac{r}{\alpha}\right)^{\frac{1}{2}}\mathrm{e}^{\alpha r}E_{\frac{1}{2}}(\alpha r)\right]^{-1}. \tag{10.27}$$

To correspond with the form (10.25) as $\alpha \to 0$ we need

$$C = [(\gamma+1)\,\omega/2a_0^2\,\alpha^{\frac{1}{2}}]\,E_{\frac{1}{2}}(\alpha r_0), \tag{10.28}$$

although this then gives a u_1 equal to $(\frac{1}{2})$ that in (10.25); and then for $\alpha r \gg 1$ we have

$$u_1 \sim \left(\frac{r_0}{r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha r}\left(\frac{a_0}{\gamma+1}\right)\left(\frac{a_0}{\omega r_0}\right)2\,\frac{(\alpha r_0)^{\frac{1}{2}}}{E_{\frac{1}{2}}(\alpha r_0)} \tag{10.29}$$

as the old-age behaviour in Rudnick's model. This has precisely the structure of the exact result (10.22), but has exactly $(\frac{1}{2})$ the value of (10.22).

For the S.M.B. method we equate $(\mathrm{d}/\mathrm{d}r)\ln[(r/r_0)^{\frac{1}{2}}u_1]$ from (10.25), for large r, to its linear counterpart $(-\alpha)$ and get

$$r_{\max} = (2\alpha)^{-1} \tag{10.30}$$

as the range for onset of old age. Then, taking linear decay from range $r_{\max}$ we have

$$\begin{aligned} u_1 &\sim \left(\frac{2a_0^2}{(\gamma+1)\,\omega r_{\max}}\right)\left(\frac{r_{\max}}{r}\right)^{\frac{1}{2}}\exp\{-\alpha(r-r_{\max})\} \\ &= \left(\frac{\pi \mathrm{e}}{2}\right)^{\frac{1}{2}}\left(\frac{1}{\pi\alpha r}\right)^{\frac{1}{2}}\mathrm{e}^{-\alpha r}\left(\frac{4a_0^2\,\alpha}{(\gamma+1)\,\omega}\right). \end{aligned} \tag{10.31}$$

This has nothing like the structure of the exact or Rudnick solutions, although it does have the right form for the $\alpha r_0 \ll 1$ limit, differing from the exact result by the curious factor $(\frac{1}{2}\pi\,\mathrm{e})^{\frac{1}{2}}$.

For thermoviscous fluids there is no linearizing transformation, and no exact result for old age with which to compare these approximate results (and exact results for constant α). There are, however, two asymptotic results. One was given by C.S. (equation 5.17) for sinusoidal cylindrical waves in the saturation amplitude conditions discussed in §9 of this paper; in those conditions one lets the diffusivity parameter $\epsilon \to 0$ while holding $\epsilon z_0 = \alpha r_0$ finite, to simulate experiments in which the source amplitude u_0 is increased in a slightly diffusive medium, with all other parameters held fixed. This results in

$$u_1 \sim \frac{a_0^2}{(\gamma+1)\,\omega(rr_0)^{\frac{1}{2}}}D_1(\alpha r_0)\exp(-\alpha r), \tag{10.32}$$

where $D_1(\alpha r_0)$ is a function which, we allege, cannot be found rationally until general exact solutions of the cylindrical Burgers' equation are known. Equation (10.22) is of precisely this form, for constant α, with

$$D_1(\alpha r_0) \equiv 4(\alpha r_0)^{\frac{1}{2}}/E_{\frac{1}{2}}(\alpha r_0), \tag{10.33}$$

whereas (10.29) and (10.31) are also of the form (10.32) but with different functions D_1.

If, on the other hand, we let $\epsilon \to 0$ with fixed z_0, we have a case dealt with in the present paper. A gross non-uniformity takes place in the weak-shock theory predictions and, as shown by (7.4) with $\lambda = 1$, the old-age decay takes the form

$$u_1 \sim C(z_0)\, u_0(\alpha r)^{-\frac{1}{2}} \exp(-\alpha r), \tag{10.34}$$

where $C(z_0)$ is an undetermined function of z_0 alone (and which, again, seems in principle to need the general solution of the cylindrical Burgers' equation for its determination). If this were to hold also for the constant α case one would need $C(z_0)$ proportional to z_0^{-1} (or perhaps asymptotic to it as $z_0 \to \infty$), in which case (10.34) would have the form

$$u_1 \sim \frac{C}{(\gamma+1)} \frac{a_0^2}{\omega(rr_0)^{\frac{1}{2}}} \frac{1}{(\alpha r_0)^{\frac{1}{2}}} \exp(-\alpha r) \tag{10.35}$$

for some constant C, and this leads to a contradiction in that it is not, in respect of its dependence on α and r_0, of the form (10.22). This, of course, is not to say that (10.34) is wrong, merely that it is not compatible with the old-age solution for constant α, which in turn is not surprising, as (10.34) represents the outcome of shock breakdown and broadening while in the constant α case the wave has a frozen-sawtooth form (aside from linear geometrical effects). It does mean, however, that in the case of cylindrical waves the Rudnick and S.M.B. predictions have not been put to a very severe test; the prediction (10.32) is not all that exacting, beyond implying amplitude saturation, and that in any case is built in to the Rudnick and S.M.B. models.

(c) *Spherical waves*

The same programme can be followed for spherical waves. Plane waves are recovered from (10.1), with $j = 2$ and constant α, if $v = r\,e^{\alpha r}u$ and

$$\xi = \int_{r_0}^{r} \frac{e^{-\alpha s}}{s}\, ds = E_1(\alpha r_0) - E_1(\alpha r), \tag{10.36}$$

where $E_1(z)$ is the usual exponential integral,

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt.$$

The sawtooth form of u is

$$u \sim \left(\frac{r_0}{r}\right) e^{-\alpha r} \frac{2a_0^2}{(\gamma+1)\, r_0} \frac{1}{E_1(\alpha r_0)} \left(\frac{\pi}{\omega} - \tau\right) \tag{10.37}$$

and the fundamental decay in old age is given by

$$u_1 \sim \left(\frac{r_0}{r}\right) e^{-\alpha r} \left(\frac{a_0^2}{(\gamma+1)\, \omega r_0}\right) \frac{4}{E_1(\alpha r_0)}. \tag{10.38}$$

For the Rudnick model, the fundamental Fourier coefficient associated with the sawtooth wave is found to be

$$q_1 = \left(\frac{r}{r_0}\right) u_1 = \frac{4a_0^2}{(\gamma+1)\,\omega r_0 \ln(r/r_0)} \tag{10.39}$$

(note that this cannot be obtained by letting $\alpha \to 0$ in (10.37), for in that equation we have already held α fixed and let $r \to \infty$ and the double limit is not commutative). Then the differential equation for q_1 is readily obtained by summing the linear and nonlinear contributions to $\mathrm{d}q_1/\mathrm{d}r$, and when this equation is solved and matched (as in the previous plane and cylindrical cases) to the solution (10.39) as αr and $\alpha r_0 \to 0$, one finds

$$u_1 = \left(\frac{r_0}{r}\right) \mathrm{e}^{-\alpha r} \left(\frac{a_0^2}{(\gamma+1)\,\omega r_0}\right) \frac{4}{[E_1(\alpha r_0) - E_1(\alpha r)]}. \tag{10.40}$$

When $\alpha r \gg 1$ this has precisely the form (10.38) of the exact solution for the constant α case. Observe that, in the respective limits $\alpha r_0 \to 0$ and $\alpha r_0 \to \infty$, (10.38) has the asymptotic forms

$$u_1 \sim \left(\frac{r_0}{r}\right) \mathrm{e}^{-\alpha r} \left(\frac{a_0^2}{(\gamma+1)\,\omega r_0}\right) \left(\frac{4}{-\ln(\alpha r_0)}\right) \tag{10.41}$$

and

$$u_1 \sim \left(\frac{r_0}{r}\right) \mathrm{e}^{-\alpha(r-r_0)} \left(\frac{4a_0^2\,\alpha}{(\gamma+1)\,\omega}\right), \tag{10.42}$$

the latter having the obvious physical interpretation.

Turning to the S.M.B. method, one finds from (10.39) that $r_{\max}$ is defined by

$$\alpha r_{\max} \ln(r_{\max}/r_0) = 1, \tag{10.43}$$

so that if (cf. C.S., p. 128) $\Gamma(x)$ is defined by

$$\Gamma(x) \ln\{\Gamma(x)/x\} = 1 \tag{10.44}$$

then $\alpha r_{\max} = \Gamma(x)$ when $\alpha r_0 = x$. Then S.M.B. assert that

$$q_1 = \frac{4a_0^2}{(\gamma+1)\,\omega r_0 \ln(r_{\max}/r_0)} \exp\{-\alpha(r-r_{\max})\}$$

from which follows

$$u_1 \sim \left(\frac{r_0}{r}\right) \mathrm{e}^{-\alpha r} \left(\frac{a_0^2}{(\gamma+1)\,\omega r_0}\right) 4\Gamma(\alpha r_0) \exp \Gamma(\alpha r_0). \tag{10.45}$$

Because $\Gamma(x)$ has the asymptotic form $-1/\ln x$ as $x \to 0$, it follows that the S.M.B. prediction has precisely the form (10.41) when $\alpha r_0 \ll 1$; and as such it agrees with the Rudnick prediction and the exact result for constant α when $\alpha r_0 \ll 1$. For $\alpha r_0 \gg 1$ we have $\Gamma(\alpha r_0) = \alpha r_0 + 1 + o(1)$ and then again the S.M.B. method reproduces precisely (10.42), which is the prediction of the exact result for constant α and of the Rudnick model.

The predictions of the present paper for spherical waves were already given in C.S., section 5. In particular, taking the equivalent of the limit $\epsilon \to 0$, $z_0 = O(1)$ of §§3–7 above they obtained a result (their equation (5.5)) which can be seen to be identical with (10.42) when it is noted that $\omega r_0 u_0/a_0^2$ is to be held fixed and $\omega\delta/u_0 a_0 \to 0$. Thus, the C.S. prediction of 'supersaturation' is recovered here by the exact constant α results, by the Rudnick prediction and by that of S.M.B. whenever (10.42) holds, in the sense that, in the limit just specified, the amplitude coefficient in (10.42) actually *decreases* as u_0 increases. However, as emphasized in C.S. and in §9 above, the limit $\epsilon \to 0, z_0 = O(1)$ is inappropriate to model saturation experiments, and in the limit $\epsilon \to 0, \epsilon z_0 = O(1)$ that *is* appropriate, C.S. obtained a result (their equation (5.18))

$$u_1 \sim \frac{a_0^2}{(\gamma+1)\,\omega r} D_2(\alpha r_0) \exp(-\alpha r), \tag{10.46}$$

where D_2 is undetermined by the analysis. The S.M.B. prediction is of this form if D_2 is chosen as

$$D_2(\alpha r_0) = 4\Gamma(\alpha r_0) \exp \Gamma(\alpha r_0),$$

as is the exact constant α result (10.38) if

$$D_2(\alpha r_0) = 4/E_1(\alpha r_0).$$

Clearly then, there are considerable degrees of overlap between old-age results predicted by our rational asymptotic analysis, by the plane-wave solutions for a thermoviscous fluid and by the exact solutions for diverging waves in a constant-α medium on the one hand, and by the *ad hoc*, but simpler and physically appealing, approximate methods of Rudnick and of S.M.B. on the other. In both the case $\alpha r_0 \ll 1$ which almost invariably prevails in experiment, and when $\alpha r_0 \gg 1$, the results from all these approaches are remarkably close in functional form and even in numerical detail for spherical waves, although the results cannot really be said to be identical because it is not clear what asymptotic limit should be understood in connection with the *ad hoc* approaches. For this reason we do not attempt to pursue the similarities further. Nor do we more than mention the work of Parker (1981), who applies a logarithmic transformation of Cole–Hopf type to the Burgers' equation for spherical waves and then linearizes the transformed equation on a small amplitude basis. This procedure does not yield solutions having any overlap (in the matched expansion sense) with other approximations for spherical waves (nor does Parker claim that it does) and we have been unable to use Parker's approach to obtain any old-age predictions for comparison with other results.

On the experimental front there is very little with which one can compare predictions of old-age behaviour. The predictions of weak-shock theory which apply prior to old age have, of course, been tested many times, with some success as far as the overall wave properties are concerned (shock strength as a function of range in particular). There remain, however, difficulties in reconciling observed and predicted shock wave thicknesses when the shock propagates over a large range from an aircraft to the ground (see, for example, Johannesen & Hodgson 1979). While most attempts to reconcile them have concentrated on the extreme sensitivity of relaxation attenuation mechanisms to atmospheric conditions, another possibility is suggested by the present work. That is, that the shock evolution, as a cylindrical N-wave, has already passed beyond the stage at which a steady balance of Taylor type can be maintained when it reaches the ground, and all attempts to relate the observed thickness to one predicted from a Taylor solution must necessarily fail. (By a Taylor solution here we simply mean one in which nonlinearity and dissipation are locally in balance; the dissipation must be dominated by relaxation mechanisms, it being generally agreed that the Taylor thickness based on a thermoviscous diffusivity of sound may be smaller, by a factor of up to 10^{-4}, than the Taylor thickness governed by relaxation attenuation; cf. Johannesen & Hodgson (1979).) This possibility cannot be checked from the solutions so far obtained, however, as density stratification must surely be taken into account as at least as important as cylindrical decay. We hope to return to this issue in future work.

Two important experiments on amplitude saturation must be mentioned. These are first the studies of Shooter *et al.* (1974; and see references there to much earlier and somewhat inconclusive experimental studies) in which saturation was observed in spherical sinusoidal waves in water, and those of Webster & Blackstock (1977) (and Webster (1977) for more detail) in which saturation of plane waves propagating in air in tubes was observed. In the S.M.B. paper, source sound pressure levels of up to 135 dB re 0.1 N m^{-2} at an equivalent source radius of 3 ft† were used, with a fundamental frequency of 450 kHz. For these conditions it is expected that a shock should first form at 6 ft, that a fully developed sawtooth wave should be formed by 21 ft and that the onset of old age ($r = r_{max}$ as defined in (10.43)) lies around 135 ft. (This mixture of units is that used in the cited paper.) This behaviour was indeed observed in the experiments, and quite remarkable agreement was found between the measurements and the S.M.B. predictions discussed earlier, the predicted sound pressure levels in old age exceeding those measured by not more than 2 dB at the highest drive levels and largest ranges used. The general agreement between the S.M.B. predictions and others noted earlier makes the issue of saturation in spherical waves rather well understood, at any rate for those high frequencies at which thermoviscosity dominates the attenuation.

In the case of propagation in tubes, the dominant attenuation mechanism is associated with boundary- (Stokes-) layer dissipation by thermoviscous forces, rather than 'mainstream' dissipation by relaxation or thermoviscosity, at any rate for frequencies of the order of a few kilohertz used by Webster & Blackstock (1977). Indeed, for the conditions of their experiment, mainstream thermoviscous attenuation dominates tube-wall losses only when the frequency exceeds 2×10^5 Hz. Webster & Blackstock find that the standard Kirchhoff formula for wall losses in the boundary layers adequately describes the measured attenuation under linear propagation, this leading to $\alpha \sim \omega^{\frac{1}{2}}$ rather than $\alpha \sim \omega^2$. One might then expect the predictions of a constant-α plane-wave theory to provide a reasonable first approximation, and Webster & Blackstock conclude that (10.5) above does agree well with their measurements at the shorter of two ranges (14.8 and 25.8 m) at each of which measurements were taken and amplitude saturation clearly observed. (The tube-wall radius was 2.5 cm, the frequencies 0.5–3.57 kHz and the source sound pressure levels up to 165 dB re 2×10^{-5} N m^{-2} in air.) At the larger range, (10.5) underpredicted the amplitudes by about 2 dB at 3.57 kHz, although the agreement remained excellent at the lower frequencies 0.5 and 1.0 kHz. Webster & Blackstock could not account for this relatively minor discrepancy (it should be noted the 'attenuation' solely due to nonlinear effects is very large here, of the order of 14 dB) and neither can we. The prediction (10.5) is ascribed by Webster & Blackstock to Rudnick; we prefer to argue that the constant α model leading to (10.5) is an appropriate model, and that (10.5) is then exact, and coincides with the prediction from Rudnick's method.

11. Conclusions

We now summarize the results of §§4–9 in a descriptive way. There are distinct broad classes of long-range behaviour depending upon the asymptotic behaviour of the area function $\mathscr{A}(r)$. First we deal with sinusoidal initial signals.

† 1 ft = 0.3048 m.

For 'negative exponential' $\mathscr{A}(r)$ (defined in §4) there is no non-uniformity in the composite of lossless solution plus Taylor shocks, and in this case weak-shock theory is valid to indefinitely large ranges. For all non-exponential variation less than spherical, i.e. for $\mathscr{A}(r) \sim r^{\lambda}$ and $-\infty < \lambda < 2$, thus including all algebraic convergence and all subspherical algebraic divergence, old age *is* reached at sufficiently large ranges. It is reached immediately beyond physical ranges of order α^{-1} (α is the small-signal attenuation coefficient for the fundamental frequency), at which range a gross non-uniformity arises involving shock thickening and a simultaneous breakdown of the Taylor shock structure. Once this non-uniformity takes place, the full GBE holds over the whole wave, but reverts to the linear form at any asymptotically larger range. This class is typified by the cylindrical Burgers' equation studied in C.S. For spherical and all superspherical divergence there is a different route to old age. The non-uniformity in the 'lossless plus Taylor shocks' composite this time is localized in the shocks. Outside the shocks the flow is changing so rapidly that the Taylor balance cannot be maintained, and the shocks develop an evolutionary structure governed by the full GBE. At much larger ranges ($r = O(\alpha^{-1})$ again) the evolutionary shocks themselves undergo a thickening non-uniformity and old age ensues.

The situation for N-waves is different in some respects. For *all converging horns* weak-shock theory holds uniformly to arbitrarily large ranges. For horns of ultimately constant area a new type of non-uniformity arises. Although it is again a gross non-uniformity, it arises this time not because of shock thickening (the shocks are of fixed $O(\epsilon)$ thickness in the plane case) nor because of failure of Taylor structure due to rapid changes outside the shock but because of shock profile distortion through the cumulative effects of 'shock displacement due to diffusivity' (Lighthill 1956; C.S.). When this non-uniformity occurs we are fortunate in having the plane Burgers' equation whose exact solution can be used to analyse the situation in all detail and to follow the transition to old age.

For subexponential divergences the situation for N-waves is essentially the same as stated above for sinusoidal waves; old age follows a gross non-uniformity for subspherical divergence, while for spherical and superspherical subexponential divergence a localized non-uniformity first arises, then a gross non-uniformity. If the divergence is exponential, however, the overall N-wave length scale increases so rapidly that the evolutionary shocks which follow the localized non-uniformity never become sufficiently thick to induce the subsequent gross non-uniformity. The wave therefore continues indefinitely with lossless arcs governed by geometrical and nonlinear effects separated by a 'frozen' pair of shock waves described by error function solutions of the *linearized* GBE, this being the asymptotic form assumed by the evolutionary shocks satisfying the full GBE. In a curious sense, weak-shock theory continues to hold for this class of $\mathscr{A}(r)$; the lossless arcs are correctly predicted by weak-shock theory, as are the shock locations, but the interior structure of the shocks no longer has the Taylor balance tacitly assumed in the usual form of weak-shock theory.

It should be noted that there are finer subdivisions within the above broad classification, according to details of the scalings required; and that when the full GBE is referred to as arising following a non-uniformity, we mean a canonical form of it, with an area function corresponding to the algebraic or exponential function 'closest' to $\mathscr{A}(r)$. Further, although it is not possible to find exact solutions of that GBE, we are none the less able to find completely the leading-order behaviour at arbitrarily large ranges for spherical and all superspherical divergences, for both N-waves and sinusoidal waves. In addition, of course, we have that behaviour for plane waves, and for all cases (now delineated) in which weak-shock theory remains valid.

With regard to amplitude saturation in old age, we have shown that this is a general phenomenon, at any rate within the confines of the model equation (1.1) and in a limit in which the source Mach number u_0/a_0 is allowed to increase indefinitely for *fixed* values of geometrical and diffusive parameters $\omega r_0/a_0$ and $\delta\omega/a_0^2$. This reason behind the generality is simply that in this limit, shock formation, thickening and amplitude saturation all take place at ranges smaller than those at which geometrical effects have any significance.

We close by emphasizing the need now for direct numerical checks on the asymptotic predictions made here. For example, there are now explicit predictions of the old-age amplitude, for N-waves and sinusoidal waves, for spherical and all superspherical divergences, and these should now form the basis of tests of both the asymptotic and numerical schemes.

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Appendix I

Consider the boundary value problem defined by (2.5 and 2.6) and let $f(\theta)$ be three times differentiable. Upon setting $W = W_0(z,\theta) + \epsilon W_1(z,\theta) + o(\epsilon)$ in (2.5) we get

$$W_0(z,\theta) = f(p), \quad \text{where} \quad \theta = p - zf(p), \tag{AI 1}$$

and
$$W_1(z,\theta) = \frac{f''(p)}{1-zf'(p)}\int_0^z \frac{a(z')\,\mathrm{d}z'}{[1-z'f'(p)]^2}, \tag{AI 2}$$

p being the characteristic variable. In order to determine the value of z, z_B, beyond which W_0 is triple-valued, we must find the value of z at which $\partial W_0/\partial\theta$ first becomes infinite. By doing this we find that $z_B = \min\,(1/f'(p))$ and hence by defining u_0, the scale associated with the initial profile which we use to non-dimensionalize the velocity or pressure perturbation u, to be such that $u_0 = \max_\tau\{f'(\omega\tau)\}$ (see §2) we find that $z_B = 1$.

In order to determine the Taylor-shock solution and its $O(\epsilon)$ correction we set $\theta^* = (\theta-\theta_s(z))/\epsilon$, where $\theta_s(z)$ is the shock location, and let $W = W_0^*(z,\theta^*)+\epsilon W_1^*(z,\theta^*)+o(\epsilon)$ in (2.5) to give
$$W_0^*(z,\theta^*) = \frac{1}{2}\left(g(z)+h(z)\,\tanh\left[\frac{[\theta^*-c(z)]\,h(z)}{4a(z)}\right]\right). \tag{AI 3}$$

Here $\qquad g(z) = h^+(z)+h^-(z), h(z) = h^+(z)-h^-(z)$, and $h^\pm(z) = \lim\limits_{\theta\to\theta_s^\pm} W_0(z,\theta)$,

the limit being taken as θ tends to θ_s from above and below, respectively. The function $c(z)$ is the correction to the shock location due to thermoviscous effects (Lighthill 1956; Crighton & Scott 1979). Also we have
$$\begin{aligned}W_1^*(z,\theta^*) = {} & (4h'a/h^2)\,y+(2g'a/h^2)\,(2y\tanh y+y^2\operatorname{sech}^2 y-1)-c'+(4a'/h)\\ & \times\tanh y\,\ln(\cosh y)-y+\operatorname{sech}^2 y\{y(1-\ln 2)-\tfrac{1}{2}y^2+\tfrac{1}{2}[\operatorname{diln}(1+\mathrm{e}^{2y})-\operatorname{diln}(2)]\})\\ & +k(z)\,(y\operatorname{sech}^2 y+\tanh y)+l(z)\operatorname{sech}^2 y,\end{aligned} \tag{AI 4}$$

where $y = [[\theta^*-c(z)]\,h(z)/4a(z)]$, $\operatorname{diln}(x) = -\int_1^x \ln t\,\mathrm{d}t/(t-1)$ is the dilogarithm (Abramowitz & Stegun 1964; Lewin 1958) and $k(z)$ and $l(z)$ are arbitrary functions of integration.

In order to determine $c(z)$ and $k(z)$ we asymptotically match the two term inner and outer solutions (AI 3, AI 4) and (AI 1, AI 2), and find that we must have
$$k(z) = \frac{4a'}{h}\ln 2+\frac{2h^{\pm\prime}}{h}c\pm c'\pm\frac{2ag'}{h^2}+\int_0^z \frac{\frac{1}{2}hh^{\pm\prime\prime}\mp h^{\pm\prime 2}-h^{\pm\prime}h'}{[(z-z')\,h^{\pm\prime}\pm\frac{1}{2}h)]^2\,h}g(z')\,\mathrm{d}z'. \tag{AI 5}$$

We may eliminate $k(z)$ between this pair of equations to obtain
$$c(z) = \frac{c_1}{h(z)}-\frac{1}{h(z)}\int_0^z\left[\int_0^{z'} g(z'')\,[E^+(z',z'')+E^-(z',z'')]\,\mathrm{d}z''+\frac{2ag'(z')}{h(z')}\right]\mathrm{d}z', \tag{AI 6}$$
where
$$E^\pm(z',z'') = \frac{\frac{1}{2}h(z')\,h^{\pm\prime\prime}(z')\mp h^{\pm\prime 2}(z')-h^{\pm\prime}(z')\,h'(z')}{[(z'-z'')\,h^{\pm\prime}(z')\pm\frac{1}{2}h(z')]^2}$$

and c_1 is a purely numerical constant. This constant is undetermined by this matching procedure and this points out the need for an 'embryo-shock region', in which the transition from a shock-free waveform to the continuous fully developed Taylor shock (AI 3) takes place. This constant c_1 may be determined by matching the Taylor-shock solution to the solution in such a region (see C.S. for details). Otherwise c_1 may be determined by an integral conservation technique. However, this constant is irrelevant to the long-range asymptotics of the solution, which is the main concern of this paper, and for that reason we do not consider it further.

Appendix II

We give here asymptotic expressions for the functions $A'(R)$,

$$I_A(R) \sim \int_1^R A^{-\frac{1}{2}}(R')\,\mathrm{d}R'; I_{I_A^2}(R) \sim \int_1^R I_A^{-1}(R')\,\mathrm{d}R'; \quad \text{and} \quad I_{I_A^4}(R) \sim \int_1^R I_A^{-2}(R')\,\mathrm{d}R',$$

for different area functions $A(R)$ as $R \to \infty$.

We shall require these asymptotic expressions when $A(R)$ has 'close to algebraic' behaviour, that is $A \in \mathrm{A}_\lambda$ for some real λ. More explicitly, this means that $A(R)$ is such that

$$RA'(R)/A(R) \to \lambda \quad \text{as} \quad R \to \infty.$$

Consequently we may set $A(R) = R^\lambda B(R)$, where $B(R)$ is such that

$$RB'(R)/B(R) \to 0 \quad \text{as} \quad R \to \infty, \tag{AII 1}$$

which means that $B(R)$ has 'less than algebraic' variation as $R \to \infty$.

From this expression for $A(R)$ in terms of $B(R)$ we find that

$$A'(R) = \lambda R^{\lambda-1}B(R) + R^\lambda B'(R) \sim R^{-1}A(R), \tag{AII 2}$$

for all λ, by virtue of (AII 1). Furthermore, we find that

$$I_A(R) \sim \begin{cases} RA^{-\frac{1}{2}}(R) & A \in \bigcup\limits_{\lambda<2} \mathrm{A}_\lambda \\ R(\ln R)\,A^{-\frac{1}{2}}(R) & A \in \mathrm{A}_2^D, \\ 1 & A \in \mathrm{A}_2^C \bigcup\limits_{\lambda>2} \mathrm{A}_\lambda \end{cases} \tag{AII 3}$$

$$I_{I_A^2}(R) \sim \begin{cases} A^{\frac{1}{2}}(R) & A \in \bigcup\limits_{\substack{\lambda<2\\ \lambda\neq 0}} \mathrm{A}_\lambda \\ (\ln R)\,A^{\frac{1}{2}}(R) & A \in \mathrm{A}_0, \\ (\ln R)^{-1}\,A^{\frac{1}{2}}(R) & A \in \mathrm{A}_2^D \\ R & A \in \mathrm{A}_2^C \bigcup\limits_{\lambda>2} \mathrm{A}_\lambda \end{cases} \tag{AII 4}$$

and

$$I_{I_A^4}(R) \sim \begin{cases} R^{-1}A(R) & A \in \bigcup\limits_{\substack{\lambda<2\\ \lambda\neq 1}} \mathrm{A}_\lambda \\ R^{-1}(\ln R)\,A(R) & A \in \mathrm{A}_1 \\ R^{-1}(\ln R)^{-2}\,A(R) & A \in \mathrm{A}_2^D \\ R & A \in \mathrm{A}_2^C \bigcup\limits_{\lambda>2} \mathrm{A}_\lambda \end{cases} \tag{AII 5}$$